Price Similarity, Spatial Chaining and International Comparisons of Prices and Real Incomes

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ESCoE Conference at King's College London (May 21-23, 2025)

Outline

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The International Comparisons Program (ICP)

The ICP computes purchasing-power-parity (PPP) exchange rates for participating countries from micro data in benchmark years.

The most recent benchmark years are 2005, 2011, 2017, 2021. For example, 176 countries participated fully in the 2011 round and 176 in the 2017 round.

ICP results are used in the Penn World Table, the World Bank's World Development Indicators, the IMF's World Economic Outlook, and the UN's Human Development Index (HDI).

Both Deaton (2016) in his Nobel lecture and Pinkovskiy and Sala-i-Martin (2020) describe the development of purchasing power parity (PPP)-exchange rates by the ICP as one of the greatest intellectual achievements in economic measurement.

The GEKS Method

The Gini-Eltetö-Köves-Szulc (GEKS) method starts from bilateral Fisher price indexes between all possible pairs of countries.

As soon as the comparison is extended to three or more countries, we have a whole matrix of Fisher price indexes:

$$F = \begin{pmatrix} 1 & P^F_{1,2} & \cdots & P^F_{1,I} \\ P^F_{2,1} & 1 & \cdots & P^F_{2,I} \\ \vdots & \vdots & & \vdots \\ P^F_{I,1} & P^F_{I,2} & \cdots & 1 \end{pmatrix}, \quad \text{where } P^F_{j,k} = 1/P^F_{k,j}.$$

In general, Fisher and other bilateral price indexes are intransitive. This means that

$$P^F_{j,l} \neq P^F_{j,k} \times P^F_{k,l}.$$

Multilateral methods, by contrast, generate transitive price indexes.

GEKS

GEKS is the most widely used multilateral method. It alters the intransitive Fisher indexes by the logarithmic least squares amount necessary to impose transitivity.

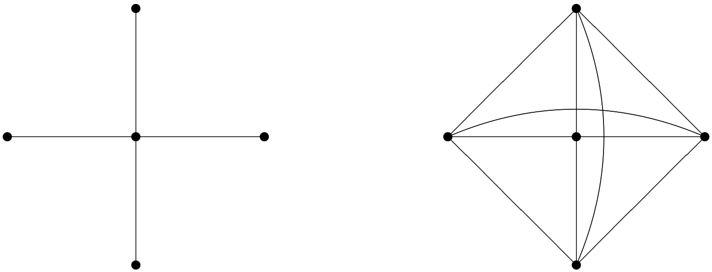
$$\min_{\ln P_j, \ln P_k} \left[\sum_{j=1}^{K} \sum_{k=1}^{K} \left(\ln P_k - \ln P_j - \ln P_{j,k}^F \right)^2 \right]$$

The solution to this minimization problem is as follows:

$$\frac{\hat{P}_k}{\hat{P}_j} = \prod_{i=1}^{K} \left(\frac{P_{i,k}^F}{P_{i,j}^F} \right)^{1/K} .$$

From a graph theory perspective, the GEKS method constructs K star spanning trees (each with a different country at the centre), and then takes the geometric mean of these K sets of results.

FIGURE 1. — EXAMPLES OF GRAPHS



Star Graph

Complete Graph

The Problem with GEKS

Some pairs of countries are very different in terms of the relative prices they face and the baskets of goods and services they consume. As a result a bilateral comparison between these country-pairs will be of poor quality.

It should be possible to improve on GEKS by either removing these weak bilateral comparisons or replacing them with spatially chained equivalents.

Choosing a Distance Metric

We consider three weighted relative price dissimilarity (WRPD) metrics proposed by Diewert (2002).

$$W1_{j,k} \equiv \sum_{n=1}^{N} \left\{ \left(\frac{s_{jn} + s_{kn}}{2} \right) \left[\left(\frac{p_{kn}}{P^F_{j,k} \times p_{jn}} - 1 \right)^2 + \left(\frac{P^F_{j,k} \times p_{jn}}{p_{kn}} - 1 \right)^2 \right] \right\}$$

$$W2_{j,k} \equiv \sum_{n=1}^{N} \left\{ \left(\frac{s_{jn} + s_{kn}}{2} \right) \left[\left(\frac{p_{kn}}{P^F_{j,k} \times p_{jn}} \right) + \left(\frac{P^F_{j,k} \times p_{jn}}{p_{kn}} \right) - 2 \right] \right\},$$

$$W3_{j,k} \equiv \sum_{n=1}^{N} \left\{ \left(\frac{s_{jn} + s_{kn}}{2} \right) \left[\ln \left(\frac{p_{kn}}{P^T_{j,k} \times p_{jn}} \right) \right]^2 \right\},$$

s_{jn} is an expenditure share, $P^F_{j,k}$ and $P^T_{j,k}$ are Fisher and Törnqvist indexes.

Another possible metric is the Laspeyres-Paasche spread:

$$LPS_{j,k} = \left| \ln \left(\frac{P^L_{j,k}}{P^P_{j,k}} \right) \right|.$$

Minimum Spanning Trees (MSTs)

Each country can be interpreted as a vertex and each Fisher bilateral comparison as an edge connecting two vertices. Each edge has an associated weight (given for example by one of Diewert's WRPD metrics).

An MST can be computed using Kruskal's algorithm as follows:

• Select the edge with the smallest weight and include it in the graph, subject to the constraint that its inclusion does not create a cycle.
• Repeat until it is no longer possible to select any more edges without creating cycles.

Examples of MSTs for ICP 2011 and 2017 based on the W3 metric for a sample of 14 countries are provided on the next slides.

Figure 1: Minimum Spanning Tree – 14 Country (W3 2011)

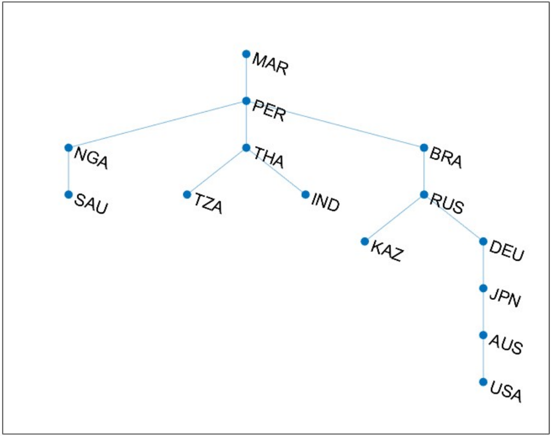
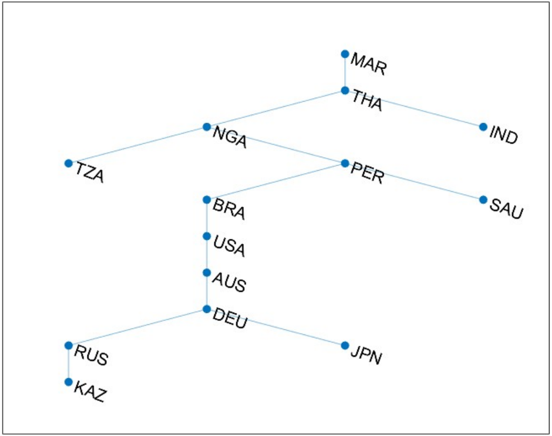


Figure 2: Minimum Spanning Tree – 14 Country (W3 2017)



Problems with MSTs

• In a comparison between K countries GEKS uses all $K(K-1)/2$ bilateral Fisher indexes. The MST uses only $K-1$. Something in between these two extremes might be preferable.

• The MST is not robust to changes in the weights metric, slight changes in the data, or from one benchmark to the next. This lack of robustness could be a source of noise in ICP if MSTs were used.

6 of the 15 edges are the same in 2011 as in 2017.

Shortest Paths

Let $SP_{b,k}$ denote the sum of the distances on the shortest path between countries b and k (based on distance metric D).

$$SP_{b,k} = \min_{j,l,\ldots,m}\{D_{b,j} + D_{j,l} + \cdots + D_{m,k}\}.$$

When LPS is used as D (and all bilaterals where $P > L$ are deleted), $SP_{b,k}$ equals the minimum chained LPS.

$$SP_{b,k} = \min_{j,l,\ldots,m}\{LPS_{b,j} + LPS_{j,l} + \cdots + LPS_{m,k}\}$$

$$= \min_{j,l,\ldots,m}\left|\ln\left(\frac{P^L_{b,j}}{P^P_{b,j}} \times \frac{P^L_{j,l}}{P^P_{j,l}} \times \cdots \times \frac{P^L_{m,k}}{P^P_{m,k}}\right)\right| = \min\left[ch(LPS)_{b,k}\right].$$

The chained PLS can be interpreted as a proxy measure of price index variance.

We show that under certain conditions:

$$E[\ln(P^L_{j,k}/P^P_{j,k})] + E[\ln(P^L_{k,l}/P^P_{k,l})] < E[\ln(P^L_{j,l}/P^P_{j,l})$$

$$\Rightarrow var(\ln P_{j,k} + \ln P_{k,l}) < var(\ln P_{j,l}).$$

and

$$E[\ln(P^L_{j,k}/P^P_{j,k})] + E[\ln(P^L_{k,l}/P^P_{k,l})] < E[\ln(P^L_{j,l}/P^P_{j,l})]$$
$$\Leftrightarrow E(D^{WRPD}_{j,k}) + E(D^{WRPD}_{k,l}) < E(D^{WRPD}_{j,l}).$$

These results provide economic foundations for spatial chaining based on LPS or a WRPD distance metric.

In practice, we find that the WRPD shortest paths tend to contain less links than the LPS shortest paths.

Figure 3: Shortest Path Spanning Tree for India – 14 Country (W3 2011)

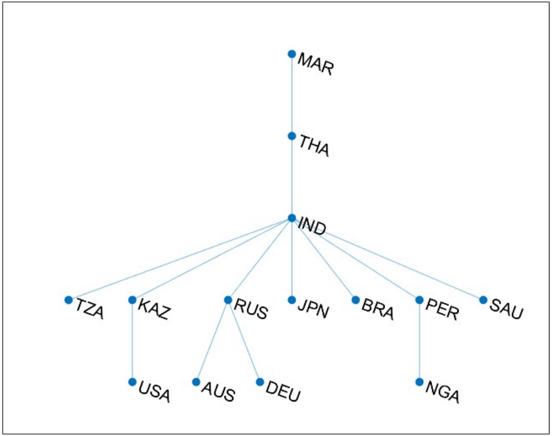


Figure 4: Shortest Path Spanning Tree for India – 14 Country (W3 2017)

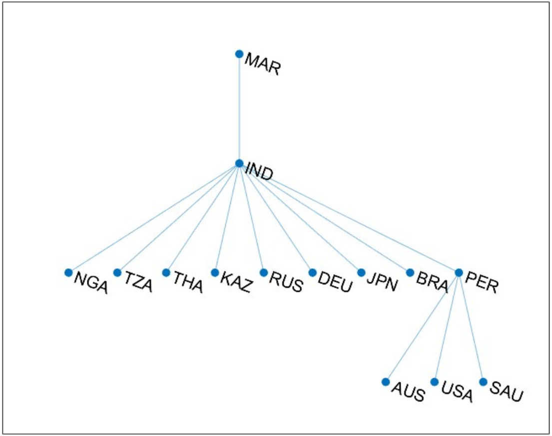


Figure 5: Shortest Path Spanning Tree for Brazil – 14 Country (W3 2011)

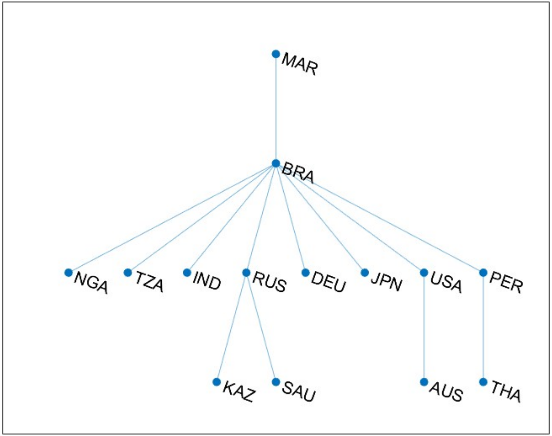
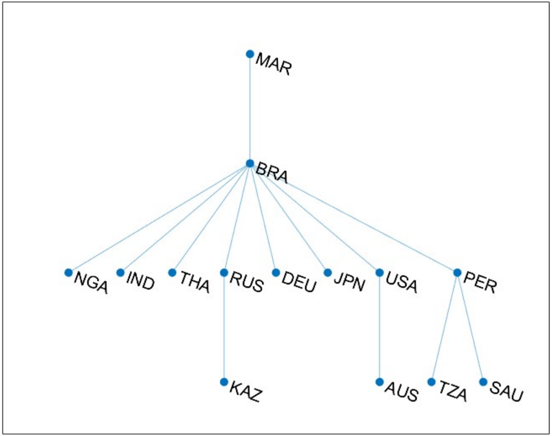


Figure 6: Shortest Path Spanning Tree for Brazil – 14 Country (W3 2017)



Shortest Path GEKS (SP-GEKS)

Each country has its own shortest path spanning tree.

7 of the 15 edges are the same for India in 2011 and 2017. 10 of the 15 edges are the same for Brazil.

SP-GKES replaces each direct Fisher index by its shortest path counterpart in the matrix of Fisher price indexes.

$$SP(F) = \begin{pmatrix} 1 & SP^F_{1,2} & \cdots & SP^F_{1,I} \\ SP^F_{2,1} & 1 & \cdots & SP^F_{2,I} \\ \vdots & \vdots & & \vdots \\ SP^F_{I,1} & SP^F_{I,2} & \cdots & 1 \end{pmatrix}$$

Then the GEKS transitivization formula is run on the matrix of shortest path Fisher indexes.

This is equivalent to taking a geometric mean of the results generated by all the shortest path spanning trees.

3. An Application to ICP 2011 and 2017

Table 1: Per Capita Consumption (US$) in 2011 and 2017

2011	MST(W3)	SP-GEKS(W3)	Fisher-US	GEKS	ICP
P. R. China	3327.8	3199.5	3458.3	3417.1	3357.2
Hong Kong	29321.7	28352.5	29818.8	29798.3	29413.5
India	2531.3	2362.3	2569.6	2528.6	2539.7
Australia	23341.2	23118.7	23341.2	24243.9	23402.2
Japan	19016.6	18495.9	19563.1	18535.2	18846.6
Luxembourg	25665.4	26151.2	25434.2	27135.5	25699.0
Ethiopia	932.7	958.1	916.6	1015.5	1056.7
2017	**MST(W3)**	**SP-GEKS(W3)**	**Fisher-US**	**GEKS**	**ICP**
P. R. China	5372.3	5137.0	5533.7	5309.4	5351.7
Hong Kong	36844.7	36603.6	37249.7	37217.9	37033.6
India	3559.1	3650.9	3895.4	3776.0	3818.5
Australia	26301.7	26234.8	27185.2	27227.8	26602.6
Japan	20322.4	20209.4	21377.1	19684.4	20344.3
Luxembourg	28257.5	28255.3	27960.8	28530.9	28006.7

Table 2: Percentage Differences in Per Capita Consumption Relative to GEKS (with USA as Base)

2011	Average Diff.	Average Absolute Diff.
Fisher	-0.12	3.91
MST(LPS)	-5.49	7.73
SP-GEKS(LPS)	5.79	6.08
MST(W1)	5.07	5.38
SP-GEKS(W1)	7.60	7.61
MST(W2)	4.43	5.19
SP-GEKS(W2)	7.82	7.82
MST(W3)	6.15	6.26
SP-GEKS(W3)	7.31	7.31
ICP	2.53	2.97

Table 3: Percentage Differences in Per Capita Consumption Relative to GEKS (with USA as Base)

2017	Average Diff.	Average Absolute Diff.
Fisher	-0.11	3.55
MST(LPS)	6.41	6.84
SP-GEKS(LPS)	7.85	8.09
MST(W1)	5.56	5.88
SP-GEKS(W1)	3.92	3.99
MST(W2)	5.39	5.74
SP-GEKS(W2)	3.83	3.89
MST(W3)	5.28	5.56
SP-GEKS(W3)	3.87	3.92
ICP	-0.69	2.19

Figure 7: Difference with GEKS as a Function of Per Capita Consumption

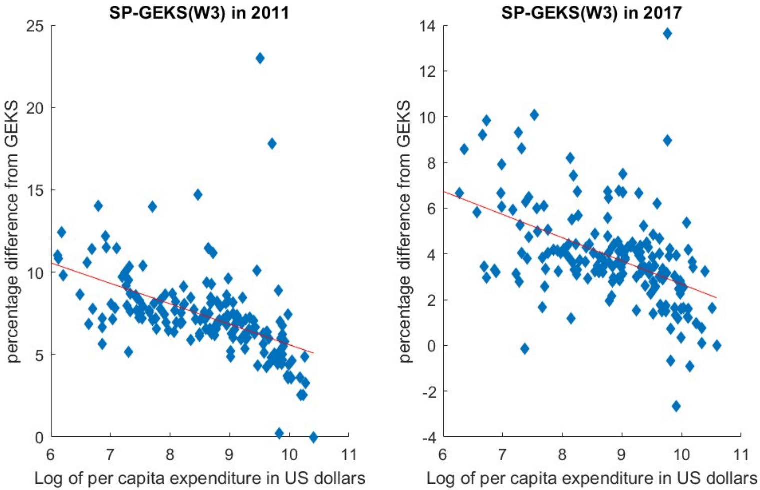


Table 4: Percentage Differences in World and Regional Consumption Relative to GEKS (with USA as Base)

2011	MST LPS	SP GEKS LPS	MST W1	SP GEKS W1	MST W2	SP GEKS W2	MST W3	SP GEKS W3	Fisher	ICP
World consump.	-5.91	2.35	3.26	4.97	3.14	5.21	3.24	4.89	0.26	1.66
Africa	-1.50	8.03	3.20	8.87	1.35	8.43	5.97	7.97	0.04	0.94
Asia Pacific	-4.21	5.34	2.40	6.83	2.59	7.07	2.09	6.67	-1.54	1.19
CIS	-3.68	4.00	7.45	6.74	7.45	6.99	7.45	6.57	2.92	4.24
EU-OECD	-7.97	0.30	2.47	2.96	2.35	3.29	2.35	3.08	1.19	1.59
Latin America	-0.88	2.81	6.00	8.28	6.02	7.88	6.18	7.36	2.61	1.81
Western Asia	0.75	5.77	12.77	12.33	12.54	12.54	12.86	11.86	-4.26	4.48
2017	MST LPS	SP GEKS LPS	MST W1	SP GEKS W1	MST W2	SP GEKS W2	MST W3	SP GEKS W3	Fisher	ICP
World consump.	3.22	4.35	2.67	2.70	2.76	2.50	2.71	2.51	-0.75	-0.51
Africa	9.04	10.36	7.35	4.99	7.07	4.16	7.27	4.06	0.12	-0.96
Asia Pacific	4.98	6.61	2.84	3.35	2.88	3.36	2.75	3.43	-3.01	-0.73
CIS	5.29	6.55	7.76	4.73	6.24	4.59	6.11	4.63	1.19	0.34
EU-OECD	1.67	2.20	2.01	1.66	1.88	1.49	1.85	1.51	0.33	-0.33
Latin America	1.11	4.65	1.42	3.30	1.34	3.23	1.12	3.16	1.68	-0.22
Western Asia	5.97	8.54	4.48	7.46	8.58	6.13	8.74	5.74	-2.81	-1.29

ICP 2011 versus Global GEKS and Spatial Chaining

The official ICP 2011 results are computed by making separate GEKS comparisons across regions, which are then combined in a way that maintains within-region fixity. So the official ICP results are not the same as a global GEKS.

SP-GEKS with any of the WRPD metrics increases global consumption in US dollars. Global inequality is reduced.

The SP-GEKS(W3) chain paths for pairs of countries in the same region mostly stay in that region.

Table 5: Within and Between Region Links in Shortest Path and Minimum s

	Total bilaterals	Shortest paths without external countries		MST paths without external countries	
2011		LPS	W3	LPS	W3
Africa	1176	507	1165	210	946
Asia Pacific	231	81	231	16	231
CIS	36	35	30	28	28
EU-OECD	1176	702	1172	420	1081
Latin America	528	318	479	115	217
Western Asia	55	20	45	9	13
2017		LPS	W3	LPS	W3
Africa	1176	459	1148	227	904
Asia Pacific	231	54	224	12	123
CIS	36	33	36	13	28
EU-OECD	1176	549	1176	271	1128
Latin America	528	83	466	16	116
Western Asia	55	23	53	10	21

4. Performance Criteria

We can count how many of the bilateral comparisons subsumed within a multilateral comparison lie within specified bounds.

We consider two sets of bounds.

(i) $P^P_{j,k} \leq M_{j,k} \leq P^L_{j,k}$

(ii) Afriat bounds:
$\max_{i,l,\ldots,m}\{P^P_{j,i}P^P_{i,l}\cdots P^P_{m,k}\} \leq M_{j,k} \leq \min_{i,l,\ldots,m}\{P^L_{j,i}P^L_{i,l}\cdots P^L_{m,k}\}$

Note: it is hard to find a set of countries for which the Afriat bounds in (ii) exist. We found sets of 28 and 22 OECD countries in 2011 and 2017, respectively, and 32 and 28 countries in Africa.

With 173 countries there are 14 878 distinct bilateral comparisons. We restrict attention to comparisons where $P^P_{j,k} < P^L_{j,k}$.

Table 6: Comparing the Performance of Multilateral Methods Using Laspeyres-Paasche Bounds and Afriat Bounds

2011	L-P Bounds	Afriat Bounds OECD (28)	Afriat Bounds Africa (32)
GEKS	13 957	233	392
MST(LPS)	11 310	261	380
SP-GEKS(LPS)	13 017	291	463
MST(W1)	13 275	216	339
SP-GEKS(W1)	13 966	309	407
MST(W2)	12 944	213	368
SP-GEKS(W2)	14 000	297	422
MST(W3)	13 230	213	368
SP-GEKS(W3)	14 026	298	421
Maximum possible	14 729	378	496

Table 7: Comparing the Performance of Multilateral Methods Using Laspeyres-Paasche Bounds and Afriat Bounds

2017	L-P Bounds	Afriat Bounds OECD (22)	Afriat Bounds Africa (28)
GEKS	13804	205	364
MST(LPS)	12212	169	367
SP-GEKS(LPS)	12391	225	377
MST(W1)	12946	208	332
SP-GEKS(W1)	13805	209	354
MST(W2)	12960	186	331
SP-GEKS(W2)	13849	214	367
MST(W3)	13022	186	331
SP-GEKS(W3)	13857	211	368
Maximum possible	14745	231	378

Main Findings

- Overall, SP-GEKS based on W3 performs best.
- W2 and W3 outperform GEKS in all cases considered. W1 outperforms GEKS in almost all cases.
- According to the L-P bounds, SP-GEKS based on LPS does not perform as well as GEKS. The implication is that it is crucial with SP-GEKS that the right distance metric is used.
- MST methods do not perform well.