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DISCUSSION PAPER

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Keywords: Consumer benefits; infrastructure services; first or der ap proximation; second order approximation; flexible functional forms; index number theory

JEL classification: C43, D61, H41, H43, H54

Kevin J. Fox, UNSW Sydney and ESCoE
k.fox@unsw.edu.au

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Consumer benefits of infrastructure services

Carmit Schwartz
UNSW Sydney and ESCoE

W. Erwin Diewert
University of British Columbia
UNSW Sydney, NBER and ESCoE

and

Kevin J. Fox*
UNSW Sydney and ESCoE

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Corresponding author: Kevin J. Fox, School of Economics & Centre for Applied Economic Research, The University of New South Wales, Sydney 2052, Australia. K.Fox@unsw.edu.au, Tel: +612 9385 3320.

1 Introduction

The number of infrastructure services provided by the public sector is great, ranging from utility services such as water, electricity and gas supplies, to communication and digital (e.g., cable, internet and telephone services) and transportation services (e.g., roads, railways and airports). Increasingly, information services are being provided through the use of Artificial Intelligence infrastructure (e.g., chatbots). Although the provision of these services benefits many households and firms, at the same time there is a substantial cost involved in providing them. Therefore, when deciding which and how much infrastructure, such as local public goods, should be provided to a given region it is important to be able to measure the benefits resulting from providing these services.

An extensive literature has mainly focused on the evaluation of the benefits of infrastructure services to the *production sector* of a country or a region; see for example Aschauer (1989), Berndt and Hansson (1992), Holtz-Eakin (1994), Seitz (1994, 1995), Morrison and Schwartz (1996), Boarnet (1998), Fernald (1999), Boisso, Grosskopf and Hayes (2000), Shanks and Barnes (2008) and Elnasri (2012). That is, by comparison, the impact of the provision of additional infrastructure services on households has been relatively unexplored. This is no doubt in part due to the complexities involved, yet an understanding of the impacts on households is key to an understanding of the political economy of public infrastructure investment; as Haughwout (2002, p. 426) notes, "residents vote and firms do not".

In this paper we provide a methodology for evaluating the benefits of infrastructure services to the *consumers* in a region. In this sense, it is in the spirit of Roback (1982), Albouy (2008), Parry (2009), Albouy (2013) and Haughwout (2002), who used a spatial general equilibrium model to assess whether consumers benefited more than firms from local price changes induced by public infrastructure. However, our approach is closer to the welfare analysis of Hicks (1940-41), Hicks (1941-42), Harberger (1971), Diewert (1992), Kanemoto (2013) and Behrens, Kanemoto and Murata (2015). In particular, we draw on the work of Diewert (1986) who developed methods to evaluate the benefits of infrastructure services to the production sector based on information on prices and quantities for the two situations that are being compared. Thus, we derive methodologies for estimating the benefits of infrastructure services to households based on observable price and quantity data and on data that can be collected through household surveys. In addition to a range of results on household benefit measures, which focus on "efficiency gains" net of redistribution effects, we derive a direct measure for pure welfare change.

The focus is to provide practical methods that can be implemented using data that can be readily collected, with the goal of expanding the range of implementable methods to assess the household benefits from infrastructure projects. Depending on the approach, the information required is different. Hence, by providing alternative approaches with different data requirements, and being specific about these requirements and underlying theoretical assumptions, we provide alternatives to practitioners and policy makers. Hence, the paper attempts to advance the understanding of theory, methods and data requirements involved in assessing consumer benefits from infrastructure services, a key area of public policy interest.

The paper proceeds as follows. In the next section we define benefit measures for consumers in the region based on a fixed price approach (i.e., we assume constant reference prices for market goods and services). In section 3 we introduce the concept of consumers' willingness to pay for infrastructure services and apply a direct approach in deriving the approximations for the benefit measures. That is, the obtained approximations are based on infrastructure services provided to the consumers in the region and on consumers' valuations of these services. This approach allows the calculation of approximations to unobservable theoretical benefit measures, provided that the required data are collected. Specifically, this approach requires that willingness to pay estimates are collected through surveys or (laboratory or online) experiments. The recent work by Brynjolfsson, Collis, and Eggers (2019) and Brynjolfsson, Collis, Diewert, Eggers and Fox (2025) on using massive online experiments may provide a way of eliciting such valuations from representative samples of the population more easily and cost effectively than in the past.[1] First order and second order approximations of household benefits are provided, with the biquadratic functional form for the household expenditure function used for the second order results. This is a flexible functional form (Diewert (1976)) and its use results in an exact relationship between theoretical benefit measures and a Bennet (1920) quantity indicator. Hence, the assumption of this functional form is relatively nonrestrictive (compared to other common functional forms, such as the Cobb-Douglas) and its use allows theoretical concepts to be estimated closely by an expression involving observable data.

In section 4 we apply an indirect approach in deriving the approximations for consumers' benefit measures. Specifically, the benefit measures are approximated using indirect information, that is, information on consumers' consumption of market goods and services and their respective prices. This removes the necessity of having estimates of households' willingness to pay for infrastructure services. However, it can be considered more demanding in terms of functional form assumptions and it will only work if *ex post* data on prices and quantities are available.

We discuss in section 5 how the benefit measures and their associated approximations can be used in quantifying the economic benefits of infrastructure services when the assumption of fixed prices is relaxed; public infrastructure may impact on land prices, and infrastructure induced urban migration may further impact on local prices. A key result here is that we find that our (first order approximation) results of section 3 can also be applied in this context to estimate household benefits.

Section 6 provides an expression, which uses data that can be readily collected, to estimate the pure welfare effect of a change in infrastructure services. Two propositions are provided, the first showing how utility change can be estimated using observable data, the second providing the conditions under which households are better off due to a change in the provision of infrastructure services. Section 7 concludes with a review of alternative approaches and results.

[1]While they focused on measuring the welfare benefits of free digital goods and used willingness to accept (forgoing consumption) valuations, the general principle could be extended to the current context. A problem is ensuring incentive compatibility in conducting such experiments in the case of willingness to pay for infrastructure services. See also related work by Diewert, Fox and Schreyer (2022) on measuring the benefits of new goods using an experimental approach.

2 Benefit Measures

In this section we define benefit measures for consumers assuming constant reference prices for market goods and services; we assume that the region under consideration is small and changes in the regional demand or supply for these goods and services do not affect the prices in other regions. By holding the prices fixed, we can focus on the pure efficiency effects of changes in the provision of infrastructure services and avoid contaminating these effects with the effects of exogenous changes in the region's terms of trade. In section 5, the prices for local market goods and services are allowed to change endogenously as the provision of infrastructure services changes and we discuss how the benefit measures developed in this section can still be used in quantifying the economic benefits of infrastructure services.

We consider a region in which there are a finite number of households, H. We assume that households in this regional economy consume two types of goods and services. The first type consists of N goods and services that households can buy at fixed positive prices $(p_1, ..., p_N)$ which we denote by the price vector p. We denote the consumption vector of these N goods and services for household h as $c^h \equiv (c_1^h, ..., c_N^h)$. We restrict the consumption vector of the N market goods and services of household h to be a nonnegative vector, i.e., $c^h \geq 0_N$ where 0_N denotes a vector of zeroes of dimension N.[2] Note that this restriction implies that the labor supply of household h is measured indirectly through its leisure consumption. For example, if household h provides l_n^h hours of labor service n then c_n^h is measured as $c_n^h \equiv 24 - l_n^h$.[3]

The second type of goods and services is a class of I infrastructure services (such as water, energy, sewage disposal, parks, airport and digital services) that are provided by all levels of government to the inhabitants of the region. Included in the list of infrastructure services are potential new services that might be provided by the government but are being provided at zero levels in the current period. The consumption by household h of the ith type of infrastructure service is denoted by the nonnegative number $S_i^h \geq 0$ for $i = 1, ..., I$ and $h = 1, ..., H$. The vector of infrastructure services utilized by household h will be denoted by the nonnegative vector $S^h \equiv (S_1^h, ..., S_I^h) \geq 0_I$ for $h = 1, ..., H$ where 0_I denotes a vector of zero of dimension I. The household may or may not be paying user fees for the use of these infrastructure services. If all of the infrastructure services were pure public goods, then we would have $S^h = S$, for $h = 1, ..., H$; i.e., each household can consume the common amount of each of the I types of infrastructure services. However, in general, we will assume that each household is utilizing a specific amount of each type of infrastructure service.

We assume that households' preferences over different combinations of the N market goods and services and of infrastructure services can be represented by utility functions, $U^h(c^h, S^h)$ for $h = 1, ..., H$, that the domain of definition for $U^h(c^h, S^h)$ is the set $V \equiv \{(c^h, S^h) : c^h \geq 0_N; S^h \geq 0_I\}$ and that $U^h(c^h, S^h)$ is continuous, non-decreasing[4] and quasi-concave in c^h. We further assume that $U^h(c^h, S^h)$ is twice continuously differentiable over its domain of definition. Note that except for differentiability we do not assume any regularity properties

[2]Notation: $y \geq 0_M$ means each component of the vector y is nonnegative, $y \gg 0_M$ means that each component is strictly positive, $y > 0_M$ means $y \geq 0_M$ but $y \neq 0_M$ and $p \cdot y$ denotes the inner product of the vectors p and y.

[3]Of course, if household h also consumes service n then this amount would be added to c_n^h.

[4]Formally, we will make the assumption that for every $(c^h, S^h) \in V$ we have $\nabla_{c^h} U^h(c^h, S^h) > 0_N$.

for $U^h(c^h, S^h)$ with respect to its S^h variable.

Under these assumptions, the restricted expenditure function of household h, denoted as e^h, is defined for $p \gg 0_N$ by minimizing the cost of achieving a given utility level $u^h > 0$, given that the household has at its disposal the vector S^h of infrastructure services. Formally, for $p \gg 0_N$ and S^h and u^h such that there exists a c^h satisfying $U^h(c^h, S^h) = u^h$ with $(c^h, S^h) \in V$, the restricted expenditure function for household h is defined by

$$e^h(u^h, p, S^h) \equiv \min_c \{p \cdot c : U^h(c, S^h) = u^h\} \quad \text{for } h = 1, ..., H. \tag{1}$$

Suppose c^h solves (1). Then we have defined the household's *restricted expenditure function* e^h by $e^h(u^h, p, S^h) = p \cdot c^h$; namely, the minimized expenditure of household h is a function of the household's given utility level u^h, the price vector p it faces for its consumption of the N market goods and services and the vector of infrastructure services S^h it has at its disposal. Note that the restricted expenditure function defined by (1) is linearly homogeneous and concave in p.

We are interested in the benefits that will accrue to household h if the government changes its infrastructure services vector from S^{h0} to S^{h1}. Before addressing this, we first note that changes in the provision of infrastructure services are likely to affect the distribution of income and welfare in the region. Therefore, when evaluating the benefits accruing to households due to changes in the provision of infrastructure services we have to be cautious to separate the redistributive effects of these changes from the efficiency effects and focus on the latter. To do that, we adopt the following approach to measuring the pure efficiency effects of changes in infrastructure services on household h: we freeze the household's utility level at its initial welfare level u^{h0} (i.e., before the changes in infrastructure services were introduced). We then ask how the minimum cost of achieving the initial welfare level will change for household h as a result of its change in infrastructure consumption. Formally, let us denote G^h as the measure of the household's gross benefits from the change in infrastructure services (S^{h0} to S^{h1}). Then we define G^h for $h = 1, ..., H$ as follows:

$$G^h(S^{h0}, S^{h1}, u^{h0}, p) \equiv -\{e^h(u^{h0}, p, S^{h1}) - e^h(u^{h0}, p, S^{h0})\}. \tag{2}$$

A few points should be mentioned regarding this gross benefit change measure. First, the benefits of the infrastructure change to household h are termed *gross* benefits because we do not net out any changes in user fees that may result from the infrastructure change. Second, G^h depends on the two infrastructure service vectors S^{h0} and S^{h1}, as well as on the household's initial level of welfare u^{h0} and the fixed price vector p. Third, the benefits to household h defined in (2) are equal to *minus* the change in the household's minimized cost of achieving its initial welfare level. That is, household h will be better off (worse off) due to changes in its infrastructure consumption if its minimized cost of achieving its initial welfare level is reduced (increased). Last, note that the household's gross benefit change measure is also a measure of gross benefit change to society since it represents a change in the household's consumption of the N market goods and services available in the region (evaluated at the constant reference prices $p \equiv (p_1, p_2, ..., p_N)$) while still maintaining its initial welfare level.

We now aggregate over all households' benefit change measures in order to define the regional gross benefit change measure due to a change in the government's provision of infrastructure services from S^{h0} to S^{h1} for household h, $h = 1, ...H$:

$$\begin{aligned} GH(S^{10}, ..., S^{H0}; S^{11}, ..., S^{H1}; u^{10}, ..., u^{H0}; p) &\equiv \sum_{h=1}^{H} G^h(S^{h0}, S^{h1}, u^{h0}, p) \\ &\equiv -\sum_{h=1}^{H} \{e^h(u^{h0}, p, S^{h1}) - e^h(u^{h0}, p, S^{h0})\}. \end{aligned} \tag{3}$$

Since prices for the N market goods and services that the region face are likely to change between period 0 and period 1 the gross benefit measure in (2) (and hence also in (3)) can be evaluated using either the prices that prevail in period 0 (denoted as p^0) or the prices that prevail in period 1 (denoted as p^1). Thus, there are two possible benefit measures for the household:

$$G^h(S^{h0}, S^{h1}, u^{h0}, p^0) \equiv -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} \tag{4}$$

and

$$G^h(S^{h0}, S^{h1}, u^{h0}, p^1) \equiv -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\}. \tag{5}$$

The above two equations represent the change in the household's consumption of the N market goods and services that is needed to maintain its initial welfare level u^{h0} due to the change in infrastructure services, where in (4) the change in the household's consumption is valued at the fixed price vector prevailing in period 0 and in (5) the change in the household's consumption is valued at the fixed price vector prevailing in period 1.

Both of (4) and (5) share a key element: they are defined in terms of differences in the households' restricted expenditure functions. Thus, in subsequent sections of this paper, we develop methods for approximating these differences using potentially observable information.

3 Approximating the Benefit Measures: A Direct Approach

In this section, to derive measures of benefit that can be calculated with (potentially) observable data, we study various first and second order approximations to the household benefit measures (4) and (5). The approach used in deriving these approximations is a direct one. That is, the benefit measures are going to be approximated using information on the infrastructure services provided to the households in the region and on the households' valuations of these services.

We begin by defining the concept of the household's "willingness to pay" for an extra unit of the ith type of infrastructure services. If a household's welfare level is u^h, then under the set of assumptions of section 2, the household would be willing to pay its reduction in expenditure on the N market goods and services that is due to an additional unit of S_i^h while still maintaining its standard of living at the utility level u^h. That is, the household should be willing to pay the following amount:

$$-\{e^h(u^h, p, S_1^h, ..., S_{i-1}^h, S_i^h + 1, S_{i+1}^h, ..., S_I^h) - e^h(u^h, p, S_1^h, ..., S_{i-1}^h, S_i^h, S_{i+1}^h, ..., S_I^h)\}. \quad (6)$$

We can approximate the difference in (6) by the partial derivative $\partial e^h(u^h, p, S^h)/\partial S_i^h$. Since this partial derivative represents the amount that a cost minimizing household is willing to pay for the use of the extra marginal unit of S_i^h, we define the household's *willingness to pay function* for marginal units of the ith infrastructure service as follows:[5]

$$W_i^h(u^h, p, S^h) \equiv -\partial e^h(u^h, p, S^h)/\partial S_i^h \qquad i = 1, ..., I. \quad (7)$$

Now, let the data for the initial situation be $p^0 \equiv (p_1^0, ..., p_N^0)$, a positive price vector for market goods and services; $u^{h0} > 0$, the household's welfare level in period 0; $c^{h0} \equiv (c_1^{h0}, ..., c_N^{h0})$, the corresponding consumption vector of household h in period 0; $S^{h0} \equiv (S_1^{h0}, ..., S_I^{h0})$, a nonnegative vector of infrastructure services that are being provided to the household h in period 0, and $W^{h0} \equiv (W_1^{h0}, ..., W_I^{h0})$, the corresponding willingness to pay vector of household h in period 0; i.e., $W_i^{h0} \equiv -\partial e^h(u^{h0}, p^0, S^{h0})/\partial S_i^h$ for $i = 1, ..., I$.

Suppose the government changes the infrastructure services vector for household h to S^{h1} in period 1. We also allow for a change in the price vector for market goods and services, so the new period 1 price vector is p^1. The household's new willingness to pay vector is defined as $W^{h1} \equiv (W_1^{h1}, ..., W_I^{h1})$ where $W_i^{h1} \equiv -\partial e^h(u^{h0}, p^1, S^{h1})/\partial S_i^h$ for $i = 1, ..., I$. Note that W_i^{h1} is evaluated at (u^{h0}, p^1, S^{h1}). That is, W_i^{h1} reflects the amount that household h, when facing period 1 prices p^1 and when having at its disposal S^{h1} infrastructure services, is willing to pay for an additional unit of S_i^h while still maintaining its standard of living at the *initial* utility level u^{h0}.[6]

We may write the two willingness to pay vectors as a gradient vector (i.e., a vector of first order partial derivatives) of the household expenditure function with respect to the components of S^h as follows:

$$W^{h0} \equiv -\nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}); \qquad W^{h1} \equiv -\nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}). \quad (8)$$

To get benefit measures that can be calculated using observable data we now develop first order approximations the household benefit measures (4) and (5).

[5]This derivation of willingness to pay functions in an expenditure function framework is due to Diewert (1986) but precursors of this concept may be found in Samuelson (1953-1954) and Diewert (1974). For further discussion on households' willingness to pay functions and their properties see Diewert (1986).

[6]W_i^{h1} is defined in terms of the initial utility level so as to focus on efficiency effects, rather than redistribution effects.

3.1 First Order Approximations: Laspeyres and Paasche type Indicators

We assume that in each period the household minimizes the cost of achieving its welfare level in that period. In particular, we assume that c^{h0} is the solution to the expenditure minimization problem (1) of household h in period 0 when $u^h = u^{h0}$, $p = p^0$ and $S^h = S^{h0}$. Thus we have the following equality:

$$e^h(u^{h0}, p^0, S^{h0}) = p^0 \cdot c^{h0} \equiv \sum_{n=1}^{N} p_n^0 c_n^{h0}. \tag{9}$$

Note that the expenditure $e^h(u^{h0}, p^0, S^{h1})$ in (4) is the hypothetical expenditure that is associated with the period 1 allocation of infrastructure services and period 0 prices and welfare level. That is, it is the expenditure that household h would have spent to achieve welfare level u^{h0} had it faced prices p^0 and had at its disposal infrastructure services S^{h1}. This expenditure is not observable but we can approximate it by means of a first order Taylor series approximation as follows:

$$\begin{aligned} e^h(u^{h0}, p^0, S^{h1}) &\simeq e^h(u^{h0}, p^0, S^{h0}) + \sum_{i=1}^{I} [\partial e^h(u^{h0}, p^0, S^{h0})/\partial S_i^h][S_i^{h1} - S_i^{h0}] \\ &= p^0 \cdot c^{h0} + \nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) \quad \text{using (9)} \\ &= p^0 \cdot c^{h0} - W^{h0} \cdot (S^{h1} - S^{h0}) \quad \text{using (8)}. \end{aligned} \tag{10}$$

Similarly, we may approximate the unobservable expenditure $e^h(u^{h0}, p^1, S^{h0})$ in (5) as follows:

$$\begin{aligned} e^h(u^{h0}, p^1, S^{h0}) &\simeq e^h(u^{h0}, p^1, S^{h1}) + \nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) \\ &= e^h(u^{h0}, p^1, S^{h1}) - W^{h1} \cdot (S^{h0} - S^{h1}) \quad \text{using (8)}. \end{aligned} \tag{11}$$

Substituting (10) and (11) into the household benefit measures (4) and (5), we obtain the following approximate benefit measures (12) and (13):

$$\begin{aligned} G^h(S^{h0}, S^{h1}, u^{h0}, p^0) &\equiv -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} \\ &\simeq -\{p^0 \cdot c^{h0} - W^{h0} \cdot (S^{h1} - S^{h0}) - p^0 \cdot c^{h0}\} \quad \text{using (9) and (10)} \\ &= W^{h0} \cdot (S^{h1} - S^{h0}) \end{aligned} \tag{12}$$

and

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^1) &\equiv -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} \\
&\simeq -\{e^h(u^{h0}, p^1, S^{h1}) - [e^h(u^{h0}, p^1, S^{h1}) - W^{h1} \cdot (S^{h0} - S^{h1})]\} \quad \text{using (11)} \\
&= W^{h1} \cdot (S^{h1} - S^{h0}). \qquad (13)
\end{aligned}
$$

Expressions (12) and (13) have the same form as the linear approximations to equivalent variation and compensation variation, respectively, as derived by Hicks (1941-42); see also Diewert (1992, p. 568, footnote 11). Each benefit measure can be calculated simply if data are available on the change in infrastructure services, $(S^{h1} - S^{h0})$ and the *ex ante* (W^{h0}) and ex post (W^{h1}) willingness to pay.

3.2 Second Order Approximation: A Bennet Type Indicator

The first order approximations of the previous section provide a means to calculate household benefit measures using observable data. However we have a choice of using either (12) or (13). Taking an arithmetic mean of the two competing measures is an option, as we would generally regard each as being equally justifiable as a measure of benefit change.[7] Taking such an average may be regarded as somewhat arbitrary. Hence, finding a theoretical reason for such an average would be preferable. This is the task of this section.

To achieve this, we now turn to develop a second order approximation to the household benefit measures (4) and (5). To do this, we first search for a class of functional forms that can approximate to the second order households' restricted expenditure functions.

We assume that households' preferences are homothetic in the N market goods and services, conditional on any vector of infrastructure services. This assumption is equivalent to the following linear homogeneity assumption on $U^h(c^h, S^h)$:[8]

$$
U^h(\lambda c^h, S^h) = \lambda U^h(c^h, S^h) \quad \text{for all } \lambda > 0; c^h \geq 0_N; S^h \geq 0_I. \qquad (14)
$$

Next we define the *unit* (utility) *expenditure function* for household h. To do so, we first note that for $(c^h, S^h) \in V$, we have,

$$
\begin{aligned}
0 &\leq c^h \cdot \nabla_{c^h} U^h(c^h, S^h) \\
&= U^h(c^h, S^h), \qquad (15)
\end{aligned}
$$

[7]In the index number context, where ratios are used rather than the differences used here, the index number which corresponds to (12) would be a Laspeyres index, as the quantities S^0 and S^1 are both are weighted by the base period price $Wh0$. Similarly, (13) corresponds to the Paasche index. Taking the arithmetic average corresponds to the Fisher index, which is the geometric mean of the Laspeyres and Paasche indexes. Such an arithmetic mean has the form of a Bennet quantity indicator, as we will see shortly.

[8]This assumption of linear homogeneity may be considered restrictive. However, it allows the derivation of a theoretical justification of taking the arithmetic mean of (4) and (5). An alternative justification would be to appeal to the axiomatic properties of the arithmetic average; see Diewert (2005) on the properties of the Bennet quantity indicator.

where the inequality follows from our assumption that for every $(c^h, S^h) \in V$ we have $\nabla_{c^h} U^h(c^h, S^h) > 0_N$ (non-decreasing in c^h utility function) and the equality follows from our assumption of linear homogeneity (14) and Euler's Theorem on homogeneous functions.[9] Thus (15) implies that for each fixed S^h, the range of $U^h(c^h, S^h)$ as c^h varies over the non-negative orthant is the set $R \equiv \{u^h : 0 \leq u^h \leq +\infty\}$. In particular, $u^h = 1$ belongs to this range set R. Hence we can define the *unit expenditure function* for household h as follows:

$$\begin{aligned} E^h(p, S^h) &\equiv \min_c \{p \cdot c : U^h(c, S^h) = 1\} \\ &= e^h(1, p, S^h) \qquad \text{for } h = 1, ..., H, \end{aligned} \tag{16}$$

where the last equality follows from (1). Note that the household's unit expenditure function defined by (16) is linearly homogeneous and concave in p.

In deriving our results, we will find it useful to use the linear homogeneity assumption (14). We can then express the household's restricted expenditure function in terms of its unit expenditure function. Specifically, for each $u^h > 0$, $p \gg 0_N$ and S^h such that there exists a c^h satisfying $U^h(c^h, S^h) = u^h$ with $(c^h, S^h) \in V$, we have

$$\begin{aligned} e^h(u^h, p, S^h) &\equiv \min_c \{p \cdot c : U^h(c, S^h) = u^h\} \\ &= \min_c \{p \cdot c : (1/u^h) U^h(c, S^h) = 1\} \\ &= \min_c \{p \cdot c : U^h(c/u^h, S^h) = 1\} \quad \text{using linear homogeneity (14)} \\ &= \min_c \{(u^h p \cdot c)/u^h : U^h(c/u^h, S^h) = 1\} \\ &= u^h \min_z \{p \cdot z : U^h(z, S^h) = 1\} \quad \text{letting } z \equiv c/u^h \\ &= u^h E^h(p, S^h) \quad \text{using definition (16)} \end{aligned} \tag{17}$$

for $h = 1, ..., H$. The advantage of relationship (17) is that it simplifies our search for a class of functional forms that can approximate to the second order households' restricted expenditure functions. Indeed, it suffices for us to find a class of functional forms that can approximate to the second order households' unit expenditure functions in order to derive second order approximations to households' benefit measures.

Thus, consider the following (normalized) biquadratic functional form for the unit expenditure function for household h:

[9] Using Euler's Theorems on homogeneous functions, our assumption of linear homogeneity implies that for every $(c^h, S^h) \in V$ we have $c^h \cdot \nabla_{c^h} U^h(c^h, S^h) = U^h(c^h, S^h)$.

$$E^h(p,S^h) \equiv \sum_{n=1}^{N} \gamma_n p_n + (1/2) \sum_{m=1}^{N-1} \sum_{n=1}^{N-1} d_{mn} p_m p_n (p_N)^{-1} + \sum_{n=1}^{N} \sum_{i=1}^{I} f_{ni} p_n S_i^h + (1/2)(\sum_{n=1}^{N} \delta_n p_n)(\sum_{i=1}^{I} \sum_{j=1}^{I} g_{ij} S_i^h S_j^h) \quad (18)$$

which can be written in matrix notation as,

$$E^h(p,S^h) \equiv \gamma \cdot p + (1/2)(p_N)^{-1}(p' \cdot Dp') + p \cdot FS^h + (1/2)(\delta \cdot p)(S^h \cdot GS^h), \quad (19)$$

where $p' \equiv (p_1, ..., p_{N-1})$, γ is an N dimensional vector with elements γ_n, D is an $N-1$ by $N-1$ symmetric and negative semi-definite matrix with elements d_{mn},[10] F is a N by I matrix with elements f_{ni}, $\delta > 0_N$ is a nonnegative, nonzero vector of *fixed constants* with elements δ_n and G is an I by I symmetric matrix with elements g_{ij}. To keep the number of unknown parameters to a minimum, and to make the household's expenditure function linear in the unknown parameters, γ_n, d_{mn}, f_{ni} and g_{ij}, we assume that the elements δ_n of the vector δ are *known* nonnegative numbers which are not all equal to zero. The chosen values for δ_n have no impact on the properties of the expenditure function and thus can be freely chosen.

For the purpose of obtaining second order approximations to household's benefit measures we define the following normalized price and willingness to pay vectors,

$$\tilde{p}^t \equiv p^t/\delta \cdot p^t; \qquad \tilde{W}^{ht} \equiv W^{ht}/\delta \cdot p^t; \qquad t = 0, 1. \quad (20)$$

As can be seen, p^0 and W^{h0} in the definition (20) are deflated by $\delta \cdot p^0 \equiv \sum_{n=1}^{N} \delta_n p_n^0$, where δ is the nonnegative, nonzero vector of fixed constants that appears in the functional form for the unit expenditure function of the household (18). Similarly, p^1 and W^{h1} in definition (20) are deflated by $\delta \cdot p^1 \equiv \sum_{n=1}^{N} \delta_n p_n^1$ to form $\tilde{p}^1$ and $\tilde{W}^{h1}$.[11]

Using definitions (2) and (20) we can define the following analogous measures to the household's benefit measures (4) and (5) as follows:

$$G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^0) \equiv -\{e^h(u^{h0}, \tilde{p}^0, S^{h1}) - e^h(u^{h0}, \tilde{p}^0, S^{h0})\}, \quad (21)$$

and

$$G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^1) \equiv -\{e^h(u^{h0}, \tilde{p}^1, S^{h1}) - e^h(u^{h0}, \tilde{p}^1, S^{h0})\}. \quad (22)$$

The household benefit measure defined by (21) is the same as the theoretical household

[10] We require the matrix D to be negative semi-definite in order to ensure the global concavity of $E^h(p, S^h)$ in p; see Diewert and Wales (1987) and the proof of their Theorem 10.

[11] In providing results on exact indicators, Diewert and Mizobuchi (2009), pp. 352-353, noted that to eliminate the effects of general inflation between the two periods being compared, it is useful to scale the prices in each period by a fixed basket price index of the form $\delta \cdot p^t$. They recommended choosing δ so that a fixed-base Laspeyres price index is used to deflate nominal prices (footnote 34, page 368).

benefit measure (4) only here the price vector p^0 is replaced by the normalized price vector $\tilde{p}^0$. Likewise, the household benefit measure defined by (22) is the same as the theoretical household benefit measure (5) where the price vector p^1 is replaced by the normalized price vectors $\tilde{p}^1$.

We then face two alternative measures of household benefit, (21) and(22). As it is unclear which is to be preferred, a solution is to take the arithmetic mean of the two. The following proposition provides a justification for doing so.

Proposition 1 *Suppose the unit expenditure function for household h, E^h, is defined by (18). Then we have the following exact identity:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^0) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \tilde{p}^1) = (1/2)[\tilde{W}^{h0} + \tilde{W}^{h1}] \cdot [S^{h1} - S^{h0}] \quad (23)$$

Proof. See the Appendix ■

The left hand side of (23) is an average of the two theoretical household benefit measures defined by (21) and (22). The right hand side is an average of the two first order approximate benefit measures defined by (12) and (13), where the original willingness to pay vectors W^{h0} and W^{h1} are replaced with the normalized willingness to pay vectors $\tilde{W}^{h0}$ and $\tilde{W}^{h1}$. The right hand side has the form of a Bennet quantity indicator; see Bennet (1920) and Diewert (2005). This can be calculated by using observable data.

The implication of the class of functional forms defined by (18) is that this class can approximate an arbitrary twice continuously differentiable unit expenditure function $E^{h^*}(p, S^h)$ to the second order. Specifically we have the following result:

Proposition 2 *Let $p^* \gg 0_N$ and $S^{h^*} \geq 0_I$ and let a given unit expenditure function E^{h^*} be twice continuously differentiable at (p^*, S^{h^*}). Then for any given nonnegative, nonzero vector $\delta > 0_N$, there exists a E^h in the class of functions defined by (18) (where the δ which appears in (18) is the same as the given δ) such that*

$$E^h(p^*, S^{h^*}) = E^{h^*}(p^*, S^{h^*}) \quad (24)$$

$$\nabla_z E^h(p^*, S^{h^*}) = \nabla_z E^{h^*}(p^*, S^{h^*}) \quad (25)$$

$$\nabla^2_{zz} E^h(p^*, S^{h^*}) = \nabla^2_{zz} E^{h^*}(p^*, S^{h^*}) \quad (26)$$

where $z \equiv (p, S^h)$. That is, the level, all $N + I$ first order partial derivatives and all $(N + I)^2$ second order partial derivatives of E^h and E^{h^*} coincide at the point (p^*, S^{h^*}).

Proof. See the Appendix ■

Combining Propositions 1 and 2 we can conclude that for any nonnegative, nonzero vector δ, the $(1/2)[\tilde{W}^{h0} + \tilde{W}^{h1}] \cdot [S^{h1} - S^{h0}]$ will approximate the average of the household's benefit measures (21) and (22) to the *second* order. Hence, there is a strong justification for using the right hand side of (23) as a benefit measure, and this can be calculated using observable data.[12]

4 Approximating the Benefit Measures: An Indirect Approach

The approach used to derive the approximations to the household benefit measures in the previous section was a direct one in the sense that the derived approximations involved direct information, that is, information on the infrastructure services provided to the household and the household's valuation over these services. In this section we employ an indirect approach for approximating the household benefit measures, in a similar vein to Kanemoto (1980), Harris (1978) and Negishi (1972). Specifically, indirect information will be used, namely, information from the market of the N goods and services in the region.

Relative to the approach of this section, the direct approach of the previous section can be viewed as being more demanding regarding the data requirement; estimates of willingness to pay are required. However, as we will see, the indirect measure is more demanding regarding functional form assumptions.

While this approach obviates the need for knowledge of the household's willingness to pay for infrastructure services, we will see that it will only work if *ex post* data are available; we will require price and quantity data for periods 0 and 1.

As before, we denote the data for the initial situation as: $p^0 \equiv (p_1^0, ..., p_N^0)$, a positive price vector for market goods and services; $u^{h0} > 0$, the household's welfare level in period 0; $c^{h0} \equiv (c_1^{h0}, ..., c_N^{h0})$, the corresponding consumption vector of household h in period 0 and $S^{h0} \equiv (S_1^{h0}, ..., S_I^{h0})$, a nonnegative vector of infrastructure services that are being provided to household h in period 0.

We assume that in each period the household minimizes the cost of achieving its welfare level in that period. That is, we assume that c^{h0} is the solution to the expenditure minimization problem (1) of household h in period 0 when $u^h = u^{h0}$, $p = p^0$ and $S^h = S^{h0}$. Similarly, c^{h1} is assumed to be the solution to the expenditure minimization problem (1) of household h in period 1 when $u^h = u^{h1}$, $p = p^1$ and $S^h = S^{h1}$. Thus we have the following equalities:

[12] Diewert (1986) derived related expressions from the producer side to those presented in this section. When approximating the benefit measures using the direct approach, similar approximations for firms and households are obtained. The first order approximation to the firm's period 0 (period 1) benefit measure is equal to the change in infrastructure services provided to the firm multiplied by the firm's willingness to pay for infrastructure services in period 0 (period 1). The second order approximation to the average of the firm's period 0 and period 1 benefit measures, evaluated at normalized prices for market goods and services, is equal to the change in infrastructure services provided to the firm multiplied by the firms's normalized average of period 0 and period 1 willingness to pay for infrastructure services.

$$e^h(u^{h0}, p^0, S^{h0}) = p^0 \cdot c^{h0} \equiv \sum_{n=1}^{N} p_n^0 c_n^{h0}; \quad e^h(u^{h1}, p^1, S^{h1}) = p^1 \cdot c^{h1} \equiv \sum_{n=1}^{N} p_n^1 c_n^{h1} \tag{27}$$

Moreover, since $e^h(u^h, p, S^h)$ is assumed to be differentiable with respect to the components of the price vector p, then using Shephard's Lemma (Shephard (1953)),[13] we have that the household's consumption vector of market goods and services in period t, c^{ht}, is equal to the vector of first order partial derivatives of $e^h(u^{ht}, p^t, S^{ht})$ with respect to the components of p. That is,

$$c^{h0} \equiv \nabla_p e^h(u^{h0}, p^0, S^{h0}); \quad c^{h1} \equiv \nabla_p e^h(u^{h1}, p^1, S^{h1}). \tag{28}$$

With this information in hand, as in the previous section, we begin by deriving alternative first order approximations to the household benefit measures (4) and (5).

4.1 First Order Approximations

Using equations (27) and (28), we can form the following three first order Taylor series approximations, which will allow us to derive household benefit measures evaluated at either period 0 or period 1 prices:

$$\begin{aligned} e^h(u^{h0}, p^1, S^{h0}) &\simeq e^h(u^{h0}, p^0, S^{h0}) + \nabla_p e^h(u^{h0}, p^0, S^{h0}) \cdot (p^1 - p^0) \\ &= p^0 \cdot c^{h0} + c^{h0} \cdot (p^1 - p^0) \quad \text{using (27) and (28)} \\ &= p^1 \cdot c^{h0} \end{aligned} \tag{29}$$

$$\begin{aligned} e^h(u^{h0}, p^0, S^{h1}) &\simeq e^h(u^{h1}, p^1, S^{h1}) + \nabla_p e^h(u^{h1}, p^1, S^{h1}) \cdot (p^0 - p^1) + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \\ &= p^1 \cdot c^{h1} + c^{h1} \cdot (p^0 - p^1) + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \quad \text{using (27) and (28)} \\ &= p^0 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \end{aligned} \tag{30}$$

Equation (30) shows that taking $p^0 \cdot c^{h1}$ by itself does not provide a first order approximation to the expenditure required to achieve period 0 household utility while facing period 0 prices and period 1 infrastructure. If period 1 utility for the household is greater than that for period 0, so that $u^{h0} - u^{h1} < 0$, then $p^0 \cdot c^{h1}$ is too large as an approximation (given $\partial e^h(u^{h1}, p^1, S^{h1})/\partial u^h > 0$). Adjusting the first term in (30) by the second term adjusts for redistribution, or the change in expenditure due to the change in utility of household h. That is, it "nets out" the effect of utility change in providing an approximation to the expenditure needed to achieve period 0 utility with period 1 infrastructure.

[13] See also Appendix 3 in Diewert (1986).

$$
\begin{aligned}
e^h(u^{h0}, p^1, S^{h1}) &\simeq e^h(u^{h1}, p^1, S^{h1}) + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \\
&= p^1 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \quad \text{using (27)} \qquad (31)
\end{aligned}
$$

Similar to (30), equation (31) shows that taking $p^1 \cdot c^{h1}$ by itself does not provide a first order approximation to the expenditure required to achieve period 0 household utility while facing period 1 prices and period 1 infrastructure; for $u^{h0} - u^{h1} < 0$ this level of expenditure is too high to achieve utility level u^{h0}, as from (27) it is the expenditure which solves the period 1 expenditure minimization problem for the higher utility level u^{h1}. Hence the value of utility change for household h between the periods has to be "netted out", using the second term in (31) in order to provide a first order approximation to $e^h(u^{h0}, p^1, S^{h1})$. Similarly if $u^{h0} - u^{h1} > 0$, in which case $p^1 \cdot c^{h1}$ is too low to provide the first order approximation to $e^h(u^{h0}, p^1, S^{h1})$.

Now substituting (29), (30) and (31) into the benefit measures (4) and (5), and we obtain the approximate benefit measures in (32) and (33):

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^0) &\equiv -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} \\
&\simeq -\{p^0 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) - p^0 \cdot c^{h0}\} \quad \text{using (27) and (30)} \\
&= p^0 \cdot (c^{h0} - c^{h1}) - \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) \qquad (32)
\end{aligned}
$$

and

$$
\begin{aligned}
G^h(S^{h0}, S^{h1}, u^{h0}, p^1) &\equiv -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} \\
&\simeq -\{p^1 \cdot c^{h1} + \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) - p^1 \cdot c^{h0}\} \quad \text{using (29) and (31)} \\
&= p^1 \cdot (c^{h0} - c^{h1}) - \frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}). \qquad (33)
\end{aligned}
$$

The first term on the right-hand side of expressions (32) and (33) gives the household's consumption change evaluated at either period 0 or period 1 prices, respectively. The second term in each expression accounts for redistributive effects, since the use of period 1 consumption data implies the need to "net out" redistributive effects. In the absence of redistributive effects ($u^{h0} = u^{h1}$), we obtain benefit measures which depend only on prices and household's consumption of market goods and services. Importantly from a practical point of view, if the redistributive effects in the region cancel out in the aggregate, then we can evaluate these approximations using only aggregate price and consumption data for the region.

For cases where the assumption of no redistributive effects (whether at the household or at the aggregate level) is inappropriate, we can derive an explicit expression for the redistributive effects if we assume that households' preferences are homothetic in the N market goods and services, conditional on any vector of infrastructure services. Specifically, we assume that the set of assumptions used in deriving the direct second order approximations to households' benefit measures (section 3.2) holds so that a unit expenditure function exists and relation (17) holds. Then, by (17) we have $e^h(u^h, p, S^h) \equiv u^h E^h(p, S^h)$ and thus, $\frac{\partial e^h(u^h,p,S^h)}{\partial u^h} \equiv E^h(p, S^h)$. Using this relation between the restricted expenditure function and the unit expenditure function we can write the expression for the redistributive effects as:[14]

$$\begin{aligned}
\frac{\partial e^h(u^{h1}, p^1, S^{h1})}{\partial u^h}(u^{h0} - u^{h1}) &= E^h(p^1, S^{h1})(u^{h0} - u^{h1}) \\
&= u^{h1} E^h(p^1, S^{h1})(\frac{u^{h0}}{u^{h1}} - 1) \\
&= e^h(u^{h1}, p^1, S^{h1})(\frac{u^{h0}}{u^{h1}} - 1) \quad \text{using (17)} \\
&= (\frac{u^{h0} - u^{h1}}{u^{h1}})(p^1 \cdot c^{h1}) \quad \text{using (27).} \qquad (34)
\end{aligned}$$

Thus, under the assumption of homothetic preferences the redistribution effects in (33) are equal to the relative change of the household's welfare level multiplied by household's period 1 expenditure. Since the redistribution effects are expressed in terms of *relative* change (and not absolute change) of the household's welfare level, we can use data on the relative change in the household's real income as a proxy for the relative change of the household's welfare level, i.e. money metric utility scaling.[15]

In the next section, we derive a second order approximation to household benefit, which again takes the form of a Bennet quantity indicator that can be calculated using observable data.

4.2 Second Order Approximation

We now turn to derive an indirect second order approximation to the household benefit measures (4) and (5). In doing so we use the same set of assumptions on households' preferences which was used in deriving the direct second order approximations (section 3.2). Moreover, we make use of the same class of functional form for the household's unit expenditure function as in (18).

We define the following household benefit measures using definition (2):

$$G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) \equiv -\{e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p_N^0}, S^{h0})\}, \qquad (35)$$

$$G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \equiv -\{e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p_N^1}, S^{h0})\}. \qquad (36)$$

[14] Another way to derive (34) without the use of the unit expenditure function is to use the fact that under this set of assumptions e^h is linearly homogeneous in u^h.

[15] While utility is generally unobservable, household income can be observed through (national statistical office or private) surveys or tax information.

The household benefit measure defined by (35) is the same as the theoretical household benefit measure (4) only here the price vector p^0 is replaced by the normalized price vector $p^0/p_N^0 \equiv (p_1^0/p_N^0, ..., p_{N-1}^0/p_N^0, 1)$. Likewise, the household benefit measure defined by (36) is the same as the theoretical household benefit measure (5) where the price vector p^1 is replaced by the normalized price vectors $p^1/p_N^1 \equiv (p_1^1/p_N^1, ..., p_{N-1}^1/p_N^1, 1)$. We can then obtain the following result.

Proposition 3 *Suppose the unit expenditure function for household h, E^h, is defined by (18) and that relations (27) and (28) hold. Then we have the following exact identity:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \quad (37)$$

Proof. See the Appendix ■

The left hand side of (37) is an average of the two theoretical household benefit measures (35) and (36). On the right hand side of (37), the first term is the average of the two first terms in the first order approximate benefit measures (32) and (33), where the price vectors p^0 and p^1 are replaced by the normalized price vectors p^0/p_N^0 and p^1/p_N^1. The second term ("netting out" redistributive effects), is the same as the second term in the first order approximate benefit measures (32) and (33), only here p^1 is replaced by the average of the normalized prices p^0/p_N^0 and p^1/p_N^1. In the absence of redistributive effects ($u^{h0} = u^{h1}$) the average of the two theoretical household benefit measures (35) and (36) equals the household's consumption change of market goods and services (a quantity change) multiplied by the average of the normalized prices p^0/p_N^0 and p^1/p_N^1. Note that this has the form of a Bennet (1920) quantity indicator; see also Diewert (2005).[16]

As was shown in Proposition 2, the class of functional forms defined by (18) can approximate an arbitrary twice continuously differentiable unit expenditure function $E^{h^*}(p, S^h)$ to the second order. Thus, Propositions 2 and 3 imply that the right hand side of (37) is an ap-

[16] In the producer context considered by Diewert (1986), the first order approximation to the firm's period t, $t = 0, 1$, benefit measure is equal to the firm's net output change where all outputs and inputs are evaluated at period t prices. The second order approximation to the average of the firm's period 0 and period 1 benefit measures is equal to the firm's net output change where all outputs and inputs are evaluated at a normalized average of period 0 and period 1 prices. While it is possible to use information on changes in quantities of market goods and services on the production side to implicitly infer benefits of changes in infrastructure provision to the production sector, we cannot do the same on the consumer side without further adjustment; the household's consumption data in period 1 implicitly conveys information on redistributive effects which we do not wish to include as part of the benefit measure. In the absence of redistributive effects we get similar approximations for both firms and households which consist of information on changes in quantities of market goods and services only.

proximate household benefit measure which approximates the average of the two theoretical household benefit measures (35) and (36) to the *second* order.

While the indirect approach yields approximations to benefit measures that do not rely on knowledge of the consumers' willingness to pay, household utility in both periods appear in expressions (34) and (37). As utility is usually not directly observable, we can assume e.g. money metric utility scaling, or attempt to establish a result based only on directly observable price and consumption data, as follows.

Corollary 1 *Suppose the conditions stipulated in Proposition 3 hold. Then the average of the two theoretical household benefit measures (35) and (36) has the following upper bound:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \leq (1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h0} \quad (38)$$

Proof. See the Appendix ■

Note that this upper bound can be calculated with only information on prices and the household's consumption of market goods and services. Note also that we can simply aggregate over households by summing the upper bounds, leading to a benefit measure that can be used for cost-benefit analysis. Due to the need for period 1 prices in (38), it can perhaps be considered to be of the most use in the *ex post* analysis of infrastructure projects. At a minimum, this upper bound provides a limit on excessive claims about potential project benefits.

5 An Endogenous Price Approach to Benefit Measures

As changes in infrastructure services in a region tend to be large discrete changes, these changes are likely to have relatively large effects on the regional economy and, therefore, they may cause systematic (endogenous) changes in the prices of local goods and services. Land is one notable local good that can have its price changed with the introduction of infrastructure, either positively or negatively. For example, the provision of a local public school can change land rent in the area.[17] Some of these price changes could occur through migration between urban areas. Thus, holding the prices of market goods and services fixed while comparing the benefits of different infrastructure provisions might not be a good strategy since some of these prices will change endogenously as the provision of infrastructure services changes.

To accommodate for local goods and services and their price endogeneity we can make use of the setup developed by Diewert (1986). In particular, we can treat the initial N market goods and services as inter-regionally traded goods and services which are supplied to or

[17] Using changes in land rents to measure the benefits of infrastructure provision has a long established literature; see e.g. Polinski and Shavell (1975) and Pines and Weiss (1976).

demanded from the region in a perfectly elastic manner, that is, their prices are assumed to be fixed. We can then introduce into the regional economy a second class of M market goods and services, where the prices of this second class of goods and services will depend on local supply and demand conditions.

Diewert (1986) derived an endogenous price benefit measure of changes in infrastructure services provision for the region as a whole, showing that the endogenous price benefit measure lies below the constant price benefit measure evaluated at the prices of the period 0.[18] Hence, an estimate of period 0 price benefit measure for the whole region will provide an upper bound to the endogenous price benefit measure. Moreover, in Diewert's period 0 price benefit measure for the whole region, the benefits to consumers are measured by (4). Thus, to estimate consumers' benefits in the period 0 price benefit measure we can use the first order approximations, (12) and (32), which were derived using the direct approach and the indirect approach, respectively.[19]

Diewert (1986) also showed that the first order approximation to the endogenous price benefit measure around the period 0 allocation of infrastructure services coincides with the *direct approach* first order approximation to the period 0 price benefit measure for the whole region. Therefore, an estimate of the direct approach first order approximation to the period 0 price benefit measure for the whole region will provide a first order approximation to the endogenous price benefit measure. Hence, the direct approach first order approximation (12) can be used to estimate the benefit.

6 Households' Welfare and the Provision of Infrastructure Services

In the previous sections we have focussed on benefit measures in terms of reduced household expenditure. In the following result we derive an explicit expression for the household's change of welfare level from period 0 to period 1, which includes redistribution effects. The derived expression is a function of the amount of infrastructure services provided to the household, the household's valuation of these services (household's willingness to pay for infrastructure services), and the household's consumption of market goods and services and their respective prices.

Proposition 4 *Suppose the unit expenditure function for household h, E^h, is defined by (18) and that relations (27) and (28) hold with $c^{h1} \neq 0_N$. Suppose further that δ in (18) is chosen*

[18]This is reminiscent of the inverse Le Chatelier principle of Diewert (1981): "We find that the more markets that adjust to the price increase in the first market, the smaller the elasticity of derived supply becomes in the first market. We could term this an inverse Le Chatelier principle, since it acts in a manner which is seemingly opposite to the usual Le Chatelier principle (cf. **?**, pp. 36-39), which says that the more markets that we allow to adjust to the effects of a price increase in the first market, the larger is the elasticity of supply in the first market. However, the two Le Chatelier principles are not really opposed. As we shall see later, our inverse Le Chatelier principle formally contains Samuelson's Le Chatelier principle as a special case." (p. 63)

[19]The first order approximation (32) can be used together with (34) so that the redistribution effects can be identified.

so that $\delta \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] = 0$. Then the ratio of the household's welfare levels in period 0 and 1, $\frac{u^{h0}}{u^{h1}}$, is given by the following expression:

$$\frac{u^{h0}}{u^{h1}} = \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0}))}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} \tag{39}$$

Proof. See the Appendix ■

As the parameter vector δ can be freely chosen (see section 3.2), the assumption on its value is nonrestrictive. Inverting (39) obviously gives the change in the utility level going from period 0 to period 1, giving a measure of the welfare change for the household. Note that when we substitute (39) into (37) then we are back to the direct measure of section 3; we have a variant of (23), which has the form of a Bennet quantity indicator, but with a different normalization. That is, we see that the average of the two theoretical household benefit measures (35) and (36) is equal to the change in infrastructure services evaluated at the average of the willingness to pay functions of period 0 and period 1.

The following result provides sufficient conditions under which households in the region are better off, from an efficiency point of view (although not necessarily with respect to the redistribution of income), due to the government's change in the provision of infrastructure services.

Proposition 5 *Suppose the conditions stipulated in Proposition 4 hold. Suppose further that $W^{h0}, W^{h1} \geq 0_I$ and $(S^{h1} - S^{h0}) \geq 0_I$. Then the average of the two theoretical household benefit measures (35) and (36) is non-negative, i.e.,:*

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \geq 0 \tag{40}$$

Proof. See the Appendix ■

The intuition is as follows. The condition $(S^{h1} - S^{h0}) \geq 0_I$ means that the government has increased the amount of infrastructure services provided to the household. The condition $W^{h0}, W^{h1} \geq 0_I$ means that the household's willingness to pay functions are non-negative. This condition can be satisfied if we assume that the household's utility function satisfies the free disposal property; this implies that if the household gets more infrastructure services then its members cannot be made worse off.[20] Thus these two conditions combined together ensure that when the government increases the level of infrastructure services provided to the household then the household is better off from an efficiency point of view, i.e. the average of the two theoretical household benefit measures (35) and (36) is non-negative.

[20]This condition may seems inappropriate in many cases, such as the provision of a new road – some residents will benefit, some will lose out yet cannot freely dispose of the infrastructure services.

7 Conclusion

This paper has presented a range of results on methods for measuring consumer benefits from changes in infrastructure services, with an emphasis on using data that can be readily collected. The key results can be summarized as follows.

In section 3, we found that when approximating benefit measures for households using a direct approach, the first order approximation to the period t benefit measure is equal to the change in infrastructure services that are provided to the household valued at the household's willingness to pay for infrastructure services in period t. The second order approximation to the average of period 0 and period 1 benefit measures is equal to the change in infrastructure services that are provided to the household valued at the household's normalized average of period 0 and period 1 willingness to pay for infrastructure services. Thus, with access to information on willingness to pay, we have easily implementable measures of benefits from changes in the provision of infrastructure services.

In section 4 we considered an alternative approach with different data requirements. When approximating the benefit measures for households using an "indirect approach," utilizing changes in prices of marketed goods and services, the first order approximation to the household's period t benefit measure is equal to the change in the household's consumption of market goods and services evaluated at period t prices less redistributive effects. Under the assumption of homothetic preferences, the household's redistribution effects equal the relative change of the household's welfare level evaluated in terms of the household's expenditure in period 1. In the absence of more attractive alternatives, the relative change in the household's real income can be used as a proxy for the relative change of the household's welfare level.

The second order approximation to the average of the household's period 0 and period 1 benefit measures for the indirect approach consisted of two expressions; the change in the household's consumption of market goods and services evaluated at a normalized average of period 0 and period 1 prices, less redistributive effects. The expression for the redistributive effects is the same as for the first order approximations, which was derived under the homotheticity assumption, only here period 1 prices are replaced by a normalized average of the prices in period 0 and period 1. The utility level in each period features in this expression for the redistributive effects, yet utility is typically not observable. However, an upper bound to the average of the household's period 0 and period 1 benefit measures is established, which is equal to the household's consumption of market goods and services in period 0 evaluated at a normalized average of period 0 and period 1 prices. Thus, it provides an upper bound on benefits using easily available data.

In section 5, we considered the relaxation of the assumption of fixed prices, to acknowledge the possibility of endogenous changes in local prices with changes to the provision of infrastructure services. We found that our fixed price results can be used to provide first order approximations to the endogenous price benefit measures.

Finally, in section 6 we derived an expression for the relative change of the household's welfare level from period 0 to period 1 to be a function of the amount of infrastructure services provided to the household, the household's valuation of these services, as well as the household's consumption of market goods and services and their respective prices. Thus, we have an expression of welfare change that can be calculated with data that can be readily collected. Using this result, it is shown that sufficient conditions exist under which households in the region are better off, from an efficiency point of view, due to the government's change in the provision of infrastructure services.

By presenting practical methods that can be implemented with data that are either already available (price and quantity information of market goods and services) or straightforwardly can be collected (provision of infrastructure services and their valuations by households), we believe we have expanded the range of implementable methods and advanced the understanding of issues involved in assessing consumer benefits from infrastructure services, a key area of public policy interest.

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Appendix: Proofs of Propositions

Proof of Proposition 1

Under the assumptions made in the setup for the second order approximation we have,

$$e^h(u^h, p, S^h) = u^h E^h(p, S^h) \tag{A1}$$

and thus,

$$\nabla_{S^h} e^h(u^h, p, S^h) = u^h \nabla_{S^h} E^h(p, S^h) \tag{A2}$$

Now using (A1) we have,

$$\begin{aligned} -\{e^h(u^{h0}, p^0, S^{h1}) - e^h(u^{h0}, p^0, S^{h0})\} &= -\{u^{h0}E^h(p^0, S^{h1}) - u^{h0}E^h(p^0, S^{h0})\} \\ &= -u^{h0}\{E^h(p^0, S^{h1}) - E^h(p^0, S^{h0})\} \end{aligned} \tag{A3}$$

and

$$\begin{aligned} -\{e^h(u^{h0}, p^1, S^{h1}) - e^h(u^{h0}, p^1, S^{h0})\} &= -\{u^{h0}E^h(p^1, S^{h1}) - u^{h0}E^h(p^1, S^{h0})\} \\ &= -u^{h0}\{E^h(p^1, S^{h1}) - E^h(p^1, S^{h0})\} \end{aligned} \tag{A4}$$

Also, since $E^h(p, S^h)$ defined by (18) is quadratic in S^h for each fixed p, its second order Taylor series expansion will be exact. Hence, we have,

$$\begin{aligned} E^h(p^0, S^{h1}) - E^h(p^0, S^{h0}) = &\nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) \\ &+ (1/2)(S^{h1} - S^{h0}) \cdot \nabla^2_{S^h S^h} E^h(p^0, S^{h0})(S^{h1} - S^{h0}) \end{aligned} \tag{A5}$$

and

$$\begin{aligned} E^h(p^1, S^{h0}) - E^h(p^1, S^{h1}) = &\nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) \\ &+ (1/2)(S^{h0} - S^{h1}) \cdot \nabla^2_{S^h S^h} E^h(p^1, S^{h1})(S^{h0} - S^{h1}) \end{aligned} \tag{A6}$$

Moreover, by (18) we have $\nabla^2_{S^h S^h} E^h(p, S^h) = (\delta \cdot p)G$. Therefore, (A5) and (A6) can be written, respectively, as,

$$E^h(p^0,S^{h1}) - E^h(p^0,S^{h0}) = \nabla_{S^h} E^h(p^0,S^{h0}) \cdot (S^{h1}-S^{h0}) \\ + (\delta \cdot p^0)(1/2)(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0}) \quad \text{(A7)}$$

and

$$E^h(p^1,S^{h0}) - E^h(p^1,S^{h1}) = \nabla_{S^h} E^h(p^1,S^{h1}) \cdot (S^{h0}-S^{h1}) \\ + (\delta \cdot p^1)(1/2)(S^{h0}-S^{h1}) \cdot G(S^{h0}-S^{h1}) \quad \text{(A8)}$$

Re-arranging (A8) we get,

$$E^h(p^1,S^{h1}) - E^h(p^1,S^{h0}) = \nabla_{S^h} E^h(p^1,S^{h1}) \cdot (S^{h1}-S^{h0}) \\ - (\delta \cdot p^1)(1/2)(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0}) \quad \text{(A9)}$$

Substituting (A7) and (A9) in (A3) and (A4), respectively, we get,

$$- \{e^h(u^{h0},p^0,S^{h1}) - e^h(u^{h0},p^0,S^{h0})\} = -u^{h0}\{E^h(p^0,S^{h1}) - E^h(p^0,S^{h0})\} \\ = -u^{h0}\{\nabla_{s^h} E^h(p^0,S^{h0}) \cdot (S^{h1}-S^{h0}) + (\delta \cdot p^0)(1/2)(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})\} \\ = -u^{h0}\nabla_{s^h} E^h(p^0,S^{h0}) \cdot (S^{h1}-S^{h0}) - (\delta \cdot p^0)(1/2)(u^{h0})(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})\} \quad \text{(A10)}$$

and

$$- \{e^h(u^{h0},p^1,S^{h1}) - e^h(u^{h0},p^1,S^{h0})\} = -u^{h0}\{E^h(p^1,S^{h1}) - E^h(p^1,S^{h0})\} \\ = -u^{h0}\{\nabla_{s^h} E^h(p^1,S^{h1}) \cdot (S^{h1}-S^{h0}) - (\delta \cdot p^1)(1/2)(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})\} \\ = -u^{h0}\nabla_{s^h} E^h(p^1,S^{h1}) \cdot (S^{h1}-S^{h0}) + (\delta \cdot p^1)(1/2)(u^{h0})(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})\} \quad \text{(A11)}$$

Using (A2), (A10) and (A11) can be written as,

$$- \{e^h(u^{h0},p^0,S^{h1}) - e^h(u^{h0},p^0,S^{h0})\} = -\nabla_{S^h} e^h(u^{h0},p^0,S^{h0}) \cdot (S^{h1}-S^{h0}) \\ - (\delta \cdot p^0)(1/2)(u^{h0})(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})\} \quad \text{(A12)}$$

and

$$-\{e^h(u^{h0},p^1,S^{h1})-e^h(u^{h0},p^1,S^{h0})\}=-\nabla_{S^h}e^h(u^{h0},p^1,S^{h1})\cdot(S^{h1}-S^{h0}) \\ +(\delta\cdot p^1)(1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A13)}$$

Using definition (8), (A12) and (A13) can be written as,

$$-\{e^h(u^{h0},p^0,S^{h1})-e^h(u^{h0},p^0,S^{h0})\}=W^{h0}\cdot(S^{h1}-S^{h0}) \\ -(\delta\cdot p^0)(1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A14)}$$

and

$$-\{e^h(u^{h0},p^1,S^{h1})-e^h(u^{h0},p^1,S^{h0})\}=W^{h1}\cdot(S^{h1}-S^{h0}) \\ +(\delta\cdot p^1)(1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A15)}$$

Dividing both sides of (A14) by $\delta\cdot p^0\neq 0$ and both sides of (A15) by $\delta\cdot p^1\neq 0$ we get,[21]

$$-\{(\delta\cdot p^0)^{-1}e^h(u^{h0},p^0,S^{h1})-(\delta\cdot p^0)^{-1}e^h(u^{h0},p^0,S^{h0})\}=(\delta\cdot p^0)^{-1}W^{h0}\cdot(S^{h1}-S^{h0}) \\ -(1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A16)}$$

and

$$-\{(\delta\cdot p^1)^{-1}e^h(u^{h0},p^1,S^{h1})-(\delta\cdot p^1)^{-1}e^h(u^{h0},p^1,S^{h0})\}=(\delta\cdot p^1)^{-1}W^{h1}\cdot(S^{h1}-S^{h0}) \\ +(1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A17)}$$

Since $e^h(u^h,p,S^h)$ is linearly homogeneous in p (see definition (1)), (A16) and (A17) can be written as,

$$-\{e^h(u^{h0},(\delta\cdot p^0)^{-1}p^0,S^{h1})-e^h(u^{h0},(\delta\cdot p^0)^{-1}p^0,S^{h0})\}=(\delta\cdot p^0)^{-1}W^{h0}\cdot(S^{h1}-S^{h0}) \\ -(1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A18)}$$

[21] Note that $\delta>0_N$ and $p^0,p^1\gg 0_N$.

and

$$-\{e^h(u^{h0},(\delta\cdot p^1)^{-1}p^1,S^{h1})-e^h(u^{h0},(\delta\cdot p^1)^{-1}p^1,S^{h0})\}=(\delta\cdot p^1)^{-1}W^{h1}\cdot(S^{h1}-S^{h0}) + (1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A19)}$$

Using definition (20), (A18) and (A19) are equivalent to,

$$-\{e^h(u^{h0},\tilde{p}^0,S^{h1})-e^h(u^{h0},\tilde{p}^0,S^{h0})\}=\tilde{W}^{h0}\cdot(S^{h1}-S^{h0}) - (1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A20)}$$

and

$$-\{e^h(u^{h0},\tilde{p}^1,S^{h1})-e^h(u^{h0},\tilde{p}^1,S^{h0})\}=\tilde{W}^{h1}\cdot(S^{h1}-S^{h0}) + (1/2)(u^{h0})(S^{h1}-S^{h0})\cdot G(S^{h1}-S^{h0})\} \quad \text{(A21)}$$

Taking the average of the two equations (A20) and (A21) and using definitions (21) and (22) yields the desired result,

$$\begin{aligned}&(1/2)G^h(S^{h0},S^{h1},u^{h0},\tilde{p}^0)+(1/2)G^h(S^{h0},S^{h1},u^{h0},\tilde{p}^1)\equiv\\ &-(1/2)\{e^h(u^{h0},\tilde{p}^0,S^{h1})-e^h(u^{h0},\tilde{p}^0,S^{h0})\}-(1/2)\{e^h(u^{h0},\tilde{p}^1,S^{h1})-e^h(u^{h0},\tilde{p}^1,S^{h0})\}\\ &=(1/2)[\tilde{W}^{h0}+\tilde{W}^{h1}]\cdot[S^{h1}-S^{h0}]\end{aligned} \quad \text{(A22)}$$

which is (23). □

Proof of Proposition 2

Ignoring any restrictions on E^h and E^{h^*}, to solve the system of equations (24)-(26), we would require E^h to have at least $1+(N+I)+(N+I)^2$ independent parameters. However, if E^h and E^{h^*} are both twice continuously differentiable at (p^*, S^{h^*}), Young's Theorem on the symmetry of second order partial derivatives reduces the number of independent second order derivatives from $(N+I)^2$ to $N(N+1)/2+NI+I(I+1)/2$. Also, the linear homogeneity in p of E^h and E^{h^*} imply the following additional $1+N+I$ restrictions on the derivatives of E^h and E^{h^*}:

$$E^h(p^*,S^{h^*})=p^*\cdot\nabla_p E^h(p^*,S^{h^*}) \quad \text{(A23)}$$

$$[\nabla^2_{pp} E^h(p^*, S^{h^*})]p^* = 0_N \tag{A24}$$

$$[\nabla^2_{S^h p} E^h(p^*, S^{h^*})]p^* = \nabla_{S^h} E^h(p^*, S^{h^*}) \tag{A25}$$

In light of these restrictions, we see that E^h will have to have at least $N(N+1)/2 + NI + I(I+1)/2$ independent parameters.

Now let $\delta > 0_N$ be any given vector which is nonnegative and nonzero, and let us consider the functional form defined by (18) where the δ which appears in (18) is the same as the given δ. Note first that E^h defined by (18) is twice continuously differentiable and linearly homogeneous in p. Moreover, since the δ_n parameters are already determined, E^h has N independent γ_n parameters, $N(N-1)/2$ independent d_{mn} parameters, NI independent f_{ni} parameters and $I(I+1)/2$ independent g_{ij} parameters. This is the minimal number of parameters required to satisfy equations (24)-(26).

Now we are remained to find the values of the independent parameters of the functional form defined by (18) which will satisfy equations (24)-(26). To do that we first solve the following system of equations for g_{ij}, $1 \leq i \leq j \leq I$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial S_i^h \partial S_j^h = g_{ij}(\sum_{n=1}^{N} \delta_n p_n^*) = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial S_i^h \partial S_j^h \tag{A26}$$

Note that the resulting G matrix with elements g_{ij} as determined above is a symmetric matrix due to the symmetry of the second order partial derivatives of E^{h^*}. With the elements g_{ij} determined, we can then solve (A27) for f_{ni}, $n = 1, ..., N$, $i = 1, ..., I$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial p_n \partial S_i^h = f_{ni} + \delta_n \sum_{j=1}^{I} g_{ij} S_j^{h^*} = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial p_n \partial S_i^h \tag{A27}$$

Next we solve the following system of equations for d_{mn}, where $1 \leq m < n \leq N-1$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial p_m \partial p_n = (p_N^*)^{-1} d_{mn} = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial p_m \partial p_n \tag{A28}$$

Note that the symmetry of the second order partial derivatives of E^{h^*} will ensure the symmetry of the matrix D. Now using the results of (A28) we can then solve the following $N-1$ equations (A29) for d_{nn}, $n = 1, ..., N-1$:

$$\partial^2 E^h(p^*, S^{h^*})/\partial p_N \partial p_n = -(p_N^*)^{-2} \sum_{m=1}^{N-1} d_{mn} p_m^* = \partial^2 E^{h^*}(p^*, S^{h^*})/\partial p_N \partial p_n \tag{A29}$$

Finally, we can use (A30) below to solve for γ_n, $n = 1, ..., N-1$:

$$\partial E^h(p^*, S^{h^*})/\partial p_n = \gamma_n + \sum_{m=1}^{N-1} d_{mn} p_m^* (p_N^*)^{-1} + \sum_{i=1}^{I} f_{ni} S_i^{h^*} + (1/2)\delta_n \sum_{i=1}^{I}\sum_{j=1}^{I} g_{ij} S_i^{h^*} S_j^{h^*} = \partial E^{h^*}(p^*, S^{h^*})/\partial p_n \quad \text{(A30)}$$

and (A31) below to solve for γ_N:

$$\partial E^h(p^*, S^{h^*})/\partial p_N = \gamma_N - (1/2) \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} d_{mn} p_m^* p_n^* (p_N^*)^{-2} + \sum_{i=1}^{I} f_{Ni} S_i^{h^*} + (1/2)\delta_N \sum_{i=1}^{I}\sum_{j=1}^{I} g_{ij} S_i^{h^*} S_j^{h^*} = \partial E^{h^*}(p^*, S^{h^*})/\partial p_N \quad \text{(A31)}$$

Observe that all the parameters of E^h defined by (18) have now been determined. Moreover, the system of equations (24)-(26) is now fully satisfied. In particular, (A30) and (A31) together with (A23) ensure that equation (24) is satisfied. Also, (A27) and (A25) ensure that the first order partial derivatives of E^h and E^{h^*} with respect to S^h coincide. This together with equations (A30) and (A31) ensure that the system of equations (25) is satisfied. Lastly, note that (A24), (A26), (A27), (A28) and (A29) ensure that the second order partial derivatives of E^h and E^{h^*} coincide, thus satisfying the system of equations (26).[22] □

Proof of Proposition 3

Under the setup of the second order approximation we have,[23]

$$e^h(u^h, p, S^h) = u^h E^h(p, S^h) \quad \text{(A32)}$$

and thus,

$$\nabla_p e^h(u^h, p, S^h) = u^h \nabla_p E^h(p, S^h) \quad \text{(A33)}$$

Now using (A32) we have,

[22](A24), (A28) and (A29) ensure that $\nabla^2_{pp} E^h(p^*, S^{h^*}) = \nabla^2_{pp} E^{h^*}(p^*, S^{h^*})$.
[23]See equation (17).

$$- \{e^h(u^{h0}, \frac{p^0}{p^0_N}, S^{h1}) - e^h(u^{h0}, \frac{p^0}{p^0_N}, S^{h0})\} = -\{u^{h0}E^h(\frac{p^0}{p^0_N}, S^{h1}) - u^{h0}E^h(\frac{p^0}{p^0_N}, S^{h0})\}$$

$$= -u^{h0}\{E^h(\frac{p^0}{p^0_N}, S^{h1}) - E^h(\frac{p^0}{p^0_N}, S^{h0})\} \quad \text{(A34)}$$

and

$$- \{e^h(u^{h0}, \frac{p^1}{p^1_N}, S^{h1}) - e^h(u^{h0}, \frac{p^1}{p^1_N}, S^{h0})\} = -\{u^{h0}E^h(\frac{p^1}{p^1_N}, S^{h1}) - u^{h0}E^h(\frac{p^1}{p^1_N}, S^{h0})\}$$

$$= -u^{h0}\{E^h(\frac{p^1}{p^1_N}, S^{h1}) - E^h(\frac{p^1}{p^1_N}, S^{h0})\} \quad \text{(A35)}$$

If p_N is fixed, then $E^h(p, S^h)$ defined by (18) is quadratic in $p_1, ..., p_{N-1}$ and thus its second order Taylor series expansion in p will be exact for each fixed S^h. Hence we have,

$$E^h(\frac{p^1}{p^1_N}, S^{h0}) = E^h(\frac{p^0}{p^0_N}, S^{h0}) + \nabla_p E^h(\frac{p^0}{p^0_N}, S^{h0}) \cdot [(\frac{p^1}{p^1_N}) - (\frac{p^0}{p^0_N})]$$

$$+ (1/2)[(\frac{p^1}{p^1_N}) - (\frac{p^0}{p^0_N})] \cdot \nabla^2_{pp} E^h(\frac{p^0}{p^0_N}, S^{h0})[(\frac{p^1}{p^1_N}) - (\frac{p^0}{p^0_N})] \quad \text{(A36)}$$

and

$$E^h(\frac{p^0}{p^0_N}, S^{h1}) = E^h(\frac{p^1}{p^1_N}, S^{h1}) + \nabla_p E^h(\frac{p^1}{p^1_N}, S^{h1}) \cdot [(\frac{p^0}{p^0_N}) - (\frac{p^1}{p^1_N})]$$

$$+ (1/2)[(\frac{p^0}{p^0_N}) - (\frac{p^1}{p^1_N})] \cdot \nabla^2_{pp} E^h(\frac{p^1}{p^1_N}, S^{h1})[(\frac{p^0}{p^0_N}) - (\frac{p^1}{p^1_N})] \quad \text{(A37)}$$

Using definition (18), the second order partial derivatives of $E^h(p, S^h)$ with respect to prices are given by,

$$\partial^2 E^h(p, S^h)/\partial p_i \partial p_j = (\frac{1}{p_N})d_{ij} \qquad 1 \le i, j \le N-1 \quad \text{(A38)}$$

$$\partial^2 E^h(p, S^h)/\partial p_N \partial p_j = -(\frac{1}{p_N})^2 \sum_{i=1}^{N-1} d_{ij} p_i \qquad j = 1, ..., N-1 \quad \text{(A39)}$$

$$\partial^2 E^h(p,S^h)/\partial p_N \partial p_N = (\frac{1}{p_N})^3 \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i p_j \tag{A40}$$

Using (A38)-(A40), it is straight forward to evaluate the matrix of second order partial derivatives of $E^h(p,S^h)$ with respect to prices, $\nabla^2_{pp}E^h(p,S^h)$, at the points $(p^0/p^0_N, S^{h0})$ and $(p^1/p^1_N, S^{h1})$, and verify that,

$$[(\frac{p^1}{p^1_N}) - (\frac{p^0}{p^0_N})] \cdot \nabla^2_{pp}E^h(\frac{p^0}{p^0_N}, S^{h0})[(\frac{p^1}{p^1_N}) - (\frac{p^0}{p^0_N})] =$$

$$\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}[(\frac{p^1_i}{p^1_N}) - (\frac{p^0_i}{p^0_N})][(\frac{p^1_j}{p^1_N}) - (\frac{p^0_j}{p^0_N})] \equiv \phi \tag{A41}$$

and

$$[(\frac{p^0}{p^0_N}) - (\frac{p^1}{p^1_N})] \cdot \nabla^2_{pp}E^h(\frac{p^1}{p^1_N}, S^{h1})[(\frac{p^0}{p^0_N}) - (\frac{p^1}{p^1_N})] =$$

$$\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}[(\frac{p^1_i}{p^1_N}) - (\frac{p^0_i}{p^0_N})][(\frac{p^1_j}{p^1_N}) - (\frac{p^0_j}{p^0_N})] \equiv \phi \tag{A42}$$

Substituting (A41) and (A42) into (A36) and (A37), respectively, we get,

$$E^h(\frac{p^1}{p^1_N}, S^{h0}) = E^h(\frac{p^0}{p^0_N}, S^{h0}) + \nabla_p E^h(\frac{p^0}{p^0_N}, S^{h0}) \cdot [(\frac{p^1}{p^1_N}) - (\frac{p^0}{p^0_N})] + (1/2)\phi \tag{A43}$$

and

$$E^h(\frac{p^0}{p^0_N}, S^{h1}) = E^h(\frac{p^1}{p^1_N}, S^{h1}) + \nabla_p E^h(\frac{p^1}{p^1_N}, S^{h1}) \cdot [(\frac{p^0}{p^0_N}) - (\frac{p^1}{p^1_N})] + (1/2)\phi \tag{A44}$$

Now substituting (A43) into (A35) and (A44) into (A34) we have,

$$
\begin{aligned}
&-\{e^h(u^{h0},\frac{p^0}{p_N^0},S^{h1})-e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0})\}=-u^{h0}\{E^h(\frac{p^0}{p_N^0},S^{h1})-E^h(\frac{p^0}{p_N^0},S^{h0})\}\\
&\qquad=-u^{h0}\{E^h(\frac{p^1}{p_N^1},S^{h1})+\nabla_p E^h(\frac{p^1}{p_N^1},S^{h1})\cdot[(\frac{p^0}{p_N^0})-(\frac{p^1}{p_N^1})]\\
&\qquad\qquad+(1/2)\phi-E^h(\frac{p^0}{p_N^0},S^{h0})\}
\end{aligned}
\tag{A45}
$$

and

$$
\begin{aligned}
&-\{e^h(u^{h0},\frac{p^1}{p_N^1},S^{h1})-e^h(u^{h0},\frac{p^1}{p_N^1},S^{h0})\}=-u^{h0}\{E^h(\frac{p^1}{p_N^1},S^{h1})-E^h(\frac{p^1}{p_N^1},S^{h0})\}\\
&\qquad=-u^{h0}\{E^h(\frac{p^1}{p_N^1},S^{h1})-E^h(\frac{p^0}{p_N^0},S^{h0})\\
&\qquad\qquad-\nabla_p E^h(\frac{p^0}{p_N^0},S^{h0})\cdot[(\frac{p^1}{p_N^1})-(\frac{p^0}{p_N^0})]-(1/2)\phi\}
\end{aligned}
\tag{A46}
$$

Using (A32) and (A33), we can write (A45) and (A46) as,

$$
\begin{aligned}
&-\{e^h(u^{h0},\frac{p^0}{p_N^0},S^{h1})-e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0})\}=-(\frac{u^{h0}}{u^{h1}})u^{h1}E^h(\frac{p^1}{p_N^1},S^{h1})\\
&\quad-(\frac{u^{h0}}{u^{h1}})u^{h1}\nabla_p E^h(\frac{p^1}{p_N^1},S^{h1})\cdot[(\frac{p^0}{p_N^0})-(\frac{p^1}{p_N^1})]-(1/2)u^{h0}\phi+u^{h0}E^h(\frac{p^0}{p_N^0},S^{h0})\\
&\quad=-(\frac{u^{h0}}{u^{h1}})e^h(u^{h1},\frac{p^1}{p_N^1},S^{h1})-(\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1},\frac{p^1}{p_N^1},S^{h1})\cdot[(\frac{p^0}{p_N^0})-(\frac{p^1}{p_N^1})]\\
&\qquad-(1/2)u^{h0}\phi+e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0})
\end{aligned}
\tag{A47}
$$

and

$$
\begin{aligned}
-\{e^h(u^{h0},\frac{p^1}{p_N^1},S^{h1}) - e^h(u^{h0},\frac{p^1}{p_N^1},S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})u^{h1}E^h(\frac{p^1}{p_N^1},S^{h1}) \\
+ u^{h0}E^h(\frac{p^0}{p_N^0},S^{h0}) + u^{h0}\nabla_p E^h(\frac{p^0}{p_N^0},S^{h0})\cdot[(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi \\
= -(\frac{u^{h0}}{u^{h1}})e^h(u^{h1},\frac{p^1}{p_N^1},S^{h1}) + e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0}) \\
+ \nabla_p e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0})\cdot[(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] + (1/2)u^{h0}\phi \quad \text{(A48)}
\end{aligned}
$$

Since $e^h(u^h,p,S^h)$ is linearly homogeneous in p, its first order partial derivatives with respect to prices, $\partial e^h(u^h,p,S^h)/\partial p_i$ for $i = 1, ..., N$, are homogeneous functions of degree 0 in p, and therefore,

$$
e^h(u^h,\frac{p}{p_N},S^h) = (\frac{1}{p_N})e^h(u^h,p,S^h) \quad \text{(A49)}
$$

and

$$
\nabla_p e^h(u^h,\frac{p}{p_N},S^h) = \nabla_p e^h(u^h,p,S^h) \quad \text{(A50)}
$$

Using (A49) and (A50), (A47) and (A48) can be written as,

$$
\begin{aligned}
-\{e^h(u^{h0},\frac{p^0}{p_N^0},S^{h1}) - e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})e^h(u^{h1},\frac{p^1}{p_N^1},S^{h1}) \\
-(\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1},\frac{p^1}{p_N^1},S^{h1})\cdot[(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] - (1/2)u^{h0}\phi + e^h(u^{h0},\frac{p^0}{p_N^0},S^{h0}) \\
= -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p_N^1})e^h(u^{h1},p^1,S^{h1}) - (\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1},p^1,S^{h1})\cdot[(\frac{p^0}{p_N^0}) - (\frac{p^1}{p_N^1})] \\
-(1/2)u^{h0}\phi + (\frac{1}{p_N^0})e^h(u^{h0},p^0,S^{h0}) \quad \text{(A51)}
\end{aligned}
$$

and

$$
\begin{aligned}
&-\{e^h(u^{h0},\frac{p^1}{p^1_N},S^{h1})-e^h(u^{h0},\frac{p^1}{p^1_N},S^{h0})\}=-(\frac{u^{h0}}{u^{h1}})e^h(u^{h1},\frac{p^1}{p^1_N},S^{h1})\\
&\quad+e^h(u^{h0},\frac{p^0}{p^0_N},S^{h0})+\nabla_p e^h(u^{h0},\frac{p^0}{p^0_N},S^{h0})\cdot[(\frac{p^1}{p^1_N})-(\frac{p^0}{p^0_N})]+(1/2)u^{h0}\phi\\
&\quad=-(\frac{u^{h0}}{u^{h1}})(\frac{1}{p^1_N})e^h(u^{h1},p^1,S^{h1})+(\frac{1}{p^0_N})e^h(u^{h0},p^0,S^{h0})\\
&\quad\quad+\nabla_p e^h(u^{h0},p^0,S^{h0})\cdot[(\frac{p^1}{p^1_N})-(\frac{p^0}{p^0_N})]+(1/2)u^{h0}\phi
\end{aligned}
\tag{A52}
$$

As relations (27) and (28) are assumed to hold, (A51) and (A52) are equivalent to,

$$
\begin{aligned}
&-\{e^h(u^{h0},\frac{p^0}{p^0_N},S^{h1})-e^h(u^{h0},\frac{p^0}{p^0_N},S^{h0})\}=-(\frac{u^{h0}}{u^{h1}})(\frac{1}{p^1_N})e^h(u^{h1},p^1,S^{h1})\\
&\quad-(\frac{u^{h0}}{u^{h1}})\nabla_p e^h(u^{h1},p^1,S^{h1})\cdot[(\frac{p^0}{p^0_N})-(\frac{p^1}{p^1_N})]-(1/2)u^{h0}\phi+(\frac{1}{p^0_N})e^h(u^{h0},p^0,S^{h0})\\
&\quad=-(\frac{u^{h0}}{u^{h1}})(\frac{1}{p^1_N})(p^1\cdot c^{h1})-(\frac{u^{h0}}{u^{h1}})c^{h1}\cdot[(\frac{p^0}{p^0_N})-(\frac{p^1}{p^1_N})]-(1/2)u^{h0}\phi+(\frac{1}{p^0_N})(p^0\cdot c^{h0})\\
&\quad=-(\frac{u^{h0}}{u^{h1}})(\frac{p^1}{p^1_N}\cdot c^{h1})-(\frac{u^{h0}}{u^{h1}})c^{h1}\cdot[(\frac{p^0}{p^0_N})-(\frac{p^1}{p^1_N})]-(1/2)u^{h0}\phi+(\frac{p^0}{p^0_N}\cdot c^{h0})\\
&\quad=-(\frac{u^{h0}}{u^{h1}})(\frac{p^0}{p^0_N}\cdot c^{h1})-(1/2)u^{h0}\phi+(\frac{p^0}{p^0_N}\cdot c^{h0})\\
&\quad=-(\frac{p^0}{p^0_N}\cdot c^{h1})-(\frac{u^{h0}}{u^{h1}}-1)(\frac{p^0}{p^0_N}\cdot c^{h1})-(1/2)u^{h0}\phi+(\frac{p^0}{p^0_N}\cdot c^{h0})\\
&\quad=\frac{p^0}{p^0_N}\cdot[c^{h0}-c^{h1}]-(\frac{u^{h0}}{u^{h1}}-1)(\frac{p^0}{p^0_N}\cdot c^{h1})-(1/2)u^{h0}\phi
\end{aligned}
\tag{A53}
$$

and

$$
\begin{aligned}
&-\{e^h(u^{h0},\frac{p^1}{p^1_N},S^{h1})-e^h(u^{h0},\frac{p^1}{p^1_N},S^{h0})\} = -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p^1_N})e^h(u^{h1},p^1,S^{h1}) \\
&\quad +(\frac{1}{p^0_N})e^h(u^{h0},p^0,S^{h0})+\nabla_p e^h(u^{h0},p^0,S^{h0})\cdot[(\frac{p^1}{p^1_N})-(\frac{p^0}{p^0_N})]+(1/2)u^{h0}\phi \\
&= -(\frac{u^{h0}}{u^{h1}})(\frac{1}{p^1_N})(p^1\cdot c^{h1})+(\frac{1}{p^0_N})(p^0\cdot c^{h0})+c^{h0}\cdot[(\frac{p^1}{p^1_N})-(\frac{p^0}{p^0_N})]+(1/2)u^{h0}\phi \\
&= -(\frac{u^{h0}}{u^{h1}})(\frac{p^1}{p^1_N}\cdot c^{h1})+(\frac{p^0}{p^0_N}\cdot c^{h0})+c^{h0}\cdot[(\frac{p^1}{p^1_N})-(\frac{p^0}{p^0_N})]+(1/2)u^{h0}\phi \\
&= -(\frac{u^{h0}}{u^{h1}})(\frac{p^1}{p^1_N}\cdot c^{h1})+(c^{h0}\cdot\frac{p^1}{p^1_N})+(1/2)u^{h0}\phi \\
&= -(\frac{p^1}{p^1_N}\cdot c^{h1})-(\frac{u^{h0}}{u^{h1}}-1)(\frac{p^1}{p^1_N}\cdot c^{h1})+(c^{h0}\cdot\frac{p^1}{p^1_N})+(1/2)u^{h0}\phi \\
&= \frac{p^1}{p^1_N}\cdot[c^{h0}-c^{h1}]-(\frac{u^{h0}}{u^{h1}}-1)(\frac{p^1}{p^1_N}\cdot c^{h1})+(1/2)u^{h0}\phi \qquad \text{(A54)}
\end{aligned}
$$

Taking the average of the two equations (A53) and (A54) and using definitions (35) and (36) gives the desired result,

$$
(1/2)G^h(S^{h0},S^{h1},u^{h0},\frac{p^0}{p^0_N})+(1/2)G^h(S^{h0},S^{h1},u^{h0},\frac{p^1}{p^1_N}) \equiv
$$

$$
-(1/2)\{e^h(u^{h0},\frac{p^0}{p^0_N},S^{h1})-e^h(u^{h0},\frac{p^0}{p^0_N},S^{h0})\}-(1/2)\{e^h(u^{h0},\frac{p^1}{p^1_N},S^{h1})-e^h(u^{h0},\frac{p^1}{p^1_N},S^{h0})\}
$$

$$
=(1/2)[(\frac{p^0}{p^0_N})+(\frac{p^1}{p^1_N})]\cdot[c^{h0}-c^{h1}]-(1/2)(\frac{u^{h0}}{u^{h1}}-1)[(\frac{p^0}{p^0_N})+(\frac{p^1}{p^1_N})]\cdot c^{h1} \qquad \text{(A55)}
$$

which is (37). □

Proof of Corollary 1

From Proposition 3 we have,

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h0} - (1/2)(\frac{u^{h0}}{u^{h1}})[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \quad \text{(A56)}$$

Now since $(1/2)(\frac{u^{h0}}{u^{h1}})[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \geq 0$ we get,[24]

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) \leq (1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h0} \quad \text{(A57)}$$

which is (38). □

Proof of Proposition 4

Under the setup of the second order approximation we have,[25]

$$e^h(u^h, p, S^h) = u^h E^h(p, S^h) \quad \text{(A58)}$$

and thus,

$$\nabla_p e^h(u^h, p, S^h) = u^h \nabla_p E^h(p, S^h) \quad \text{(A59)}$$

Combining (A59) together with (28), which is assumed to hold, we get

$$c^{h0} = \nabla_p e^h(u^{h0}, p^0, S^{h0}) = u^{h0} \nabla_p E^h(p^0, S^{h0}) \quad \text{(A60)}$$

and

$$c^{h1} = \nabla_p e^h(u^{h1}, p^1, S^{h1}) = u^{h1} \nabla_p E^h(p^1, S^{h1}) \quad \text{(A61)}$$

To reduce clutter in notation let us re-write the class of functional forms for the household's unit expenditure function defined by (18) as follows:

[24] Note that $u^{h0}, u^{h1} > 0$; $c^{h1} \geq 0_N$; and $p \gg 0_N$.
[25] See equation (17).

$$E^h(p, S^h) \equiv \sum_{i=1}^{N} \gamma_i p_i + (1/2) \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i p_j (p_N)^{-1} + \sum_{i=1}^{N} p_i \Phi_i(S^h) \quad \text{(A62)}$$

where $\Phi_i(S^h) \equiv \sum_{j=1}^{I} f_{ij} S_j^h + \delta_i (1/2)(\sum_{j=1}^{I} \sum_{m=1}^{I} g_{jm} S_j^h S_m^h)$ and thus $\sum_{i=1}^{N} p_i \Phi_i(S^h) = \sum_{i=1}^{N} \sum_{j=1}^{I} f_{ij} p_i S_j^h + (1/2)(\sum_{i=1}^{N} \delta_i p_i)(\sum_{j=1}^{I} \sum_{m=1}^{I} g_{jm} S_j^h S_m^h)$.

Differentiating (A62) with respect to prices and making use of (A60) and (A61) yields the following equations:

$$c^{h0} = \begin{bmatrix} u^{h0}[\gamma_1 + (p_N^0)^{-1} \sum_{j=1}^{N-1} d_{1j} p_j^0 + \Phi_1(S^{h0})] \\ \vdots \\ u^{h0}[\gamma_{N-1} + (p_N^0)^{-1} \sum_{j=1}^{N-1} d_{(N-1)j} p_j^0 + \Phi_{N-1}(S^{h0})] \\ u^{h0}[\gamma_N - (1/2)(p_N^0)^{-2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^0 p_j^0 + \Phi_N(S^{h0})] \end{bmatrix} \quad \text{(A63)}$$

and

$$c^{h1} = \begin{bmatrix} u^{h1}[\gamma_1 + (p_N^1)^{-1} \sum_{j=1}^{N-1} d_{1j} p_j^1 + \Phi_1(S^{h1})] \\ \vdots \\ u^{h1}[\gamma_{N-1} + (p_N^1)^{-1} \sum_{j=1}^{N-1} d_{(N-1)j} p_j^1 + \Phi_{N-1}(S^{h1})] \\ u^{h1}[\gamma_N - (1/2)(p_N^1)^{-2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^1 p_j^1 + \Phi_N(S^{h1})] \end{bmatrix} \quad \text{(A64)}$$

Using (A63) and (A64), it is straight forward to verify that,

$$\begin{aligned} p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[&\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} d_{ij} p_i^1 p_j^0 + p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i + p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i \\ &+ p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h0}) + p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h0})] \quad \text{(A65)} \end{aligned}$$

and

$$p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1}) = u^{h1}[\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^0 + p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i + p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i$$
$$+ p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h1}) + p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h1})] \quad \text{(A66)}$$

where (A66) is derived by also using the symmetry of the D matrix.

Now using (A62), it is also straight forward to show that,

$$p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h0}) = p_N^1 E^h(p^0, S^{h0}) - p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i - (1/2)(\frac{p_N^1}{p_N^0}) \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^0 p_j^0 \quad \text{(A67)}$$

$$p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h0}) = p_N^0 E^h(p^1, S^{h0}) - p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i - (1/2)(\frac{p_N^0}{p_N^1}) \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^1 \quad \text{(A68)}$$

$$p_N^0 \sum_{i=1}^{N} p_i^1 \Phi_i(S^{h1}) = p_N^0 E^h(p^1, S^{h1}) - p_N^0 \sum_{i=1}^{N} p_i^1 \gamma_i - (1/2)(\frac{p_N^0}{p_N^1}) \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^1 \quad \text{(A69)}$$

$$p_N^1 \sum_{i=1}^{N} p_i^0 \Phi_i(S^{h1}) = p_N^1 E^h(p^0, S^{h1}) - p_N^1 \sum_{i=1}^{N} p_i^0 \gamma_i - (1/2)(\frac{p_N^1}{p_N^0}) \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^0 p_j^0 \quad \text{(A70)}$$

Substituting (A67) and (A68) into (A65) and substituting (A69) and (A70) into (A66) we get,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h0}) + p_N^0 E^h(p^1, S^{h0})] \quad \text{(A71)}$$

and

$$p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1}) = u^{h1}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})] \quad \text{(A72)}$$

where

$$\lambda \equiv \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^0 - (1/2)(\frac{p_N^1}{p_N^0})\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^0 p_j^0 - (1/2)(\frac{p_N^0}{p_N^1})\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} d_{ij}p_i^1 p_j^1 \quad \text{(A73)}$$

Now since $E^h(p, S^h)$ defined by (18) is quadratic in S^h for each fixed p, its second order Taylor series expansion will be exact. Hence, we have,

$$\begin{aligned} &E^h(p^0, S^{h1}) - E^h(p^0, S^{h0}) = \\ &\nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) + (1/2)(S^{h1} - S^{h0}) \cdot \nabla^2_{S^h S^h} E^h(p^0, S^{h0})(S^{h1} - S^{h0}) = \\ &\qquad \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) + (\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0}) \quad \text{(A74)} \end{aligned}$$

and

$$\begin{aligned} &E^h(p^1, S^{h0}) - E^h(p^1, S^{h1}) = \\ &\nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) + (1/2)(S^{h0} - S^{h1}) \cdot \nabla^2_{S^h S^h} E^h(p^1, S^{h1})(S^{h0} - S^{h1}) = \\ &\qquad \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) + (\delta \cdot p^1)(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1}) \quad \text{(A75)} \end{aligned}$$

Multiplying (A74) by p_N^1 and re-arranging yields,

$$\begin{aligned} p_N^1 E^h(p^0, S^{h0}) = p_N^1 E^h(p^0, S^{h1}) &- p_N^1 \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) \\ &- p_N^1 (\delta \cdot p^0)(1/2)(S^{h1} - S^{h0}) \cdot G(S^{h1} - S^{h0}) \quad \text{(A76)} \end{aligned}$$

Similarly, multiplying (A75) by p_N^0 and re-arranging we get,

$$\begin{aligned} p_N^0 E^h(p^1, S^{h0}) = p_N^0 E^h(p^1, S^{h1}) &+ p_N^0 \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) \\ &+ p_N^0 (\delta \cdot p^1)(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1}) \quad \text{(A77)} \end{aligned}$$

Substituting (A76) and (A77) into (A71) we obtain,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]$$
$$+ u^{h0}[p_N^0 \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) - p_N^1 \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})]$$
$$+u^{h0}[p_N^0(\delta \cdot p^1)(1/2)(S^{h0}-S^{h1}) \cdot G(S^{h0}-S^{h1}) - p_N^1(\delta \cdot p^0)(1/2)(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})] \quad \text{(A78)}$$

Note that,

$$u^{h0}[p_N^0(\delta \cdot p^1)(1/2)(S^{h0}-S^{h1}) \cdot G(S^{h0}-S^{h1}) - p_N^1(\delta \cdot p^0)(1/2)(S^{h1}-S^{h0}) \cdot G(S^{h1}-S^{h0})] =$$
$$u^{h0}(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1})[p_N^0(\delta \cdot p^1) - p_N^1(\delta \cdot p^0)] =$$
$$u^{h0}(1/2)(S^{h0} - S^{h1}) \cdot G(S^{h0} - S^{h1})(p_N^1 p_N^0)(\delta \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})]) = 0 \quad \text{(A79)}$$

where the last equality follows from the condition that $\delta \cdot [(\frac{p^1}{p_N^1}) - (\frac{p^0}{p_N^0})] = 0$.

Thus we get that,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) = u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]$$
$$+ u^{h0}[p_N^0 \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) - p_N^1 \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})] \quad \text{(A80)}$$

Furthermore,

$$u^{h0}[p_N^0 \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) - p_N^1 \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0})] =$$
$$p_N^0 u^{h0} \nabla_{S^h} E^h(p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) - p_N^1 u^{h0} \nabla_{S^h} E^h(p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) =$$
$$p_N^0 \nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}) \cdot (S^{h0} - S^{h1}) - p_N^1 \nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) =$$
$$- p_N^0 \nabla_{S^h} e^h(u^{h0}, p^1, S^{h1}) \cdot (S^{h1} - S^{h0}) - p_N^1 \nabla_{S^h} e^h(u^{h0}, p^0, S^{h0}) \cdot (S^{h1} - S^{h0}) =$$
$$p_N^0(W^{h1} \cdot (S^{h1} - S^{h0})) + p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) \quad \text{(A81)}$$

where the second equality follows from the fact that under the assumptions made in the setup for the second order approximation $e^h(u^h, p, S^h) = u^h E^h(p, S^h)$ and hence, $\nabla_{S^h} e^h(u^h, p, S^h) = u^h \nabla_{S^h} E^h(p, S^h)$. The last equality is achieved using definition (8).

Substituting (A81) into (A80) and re-arranging yields,

$$p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0})) = u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})] \tag{A82}$$

Combining (A82) together with (A72) we obtain the desired result,[26]

$$\frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0}))}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} = \frac{u^{h0}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]}{u^{h1}[\lambda + p_N^1 E^h(p^0, S^{h1}) + p_N^0 E^h(p^1, S^{h1})]} = \frac{u^{h0}}{u^{h1}} \tag{A83}$$

which is (39). □

Proof of Proposition 5

Under the conditions of Proposition 4 we have,

$$\frac{u^{h0}}{u^{h1}} = \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0}) - p_N^1(W^{h0} \cdot (S^{h1} - S^{h0})) - p_N^0(W^{h1} \cdot (S^{h1} - S^{h0}))}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} \tag{A84}$$

Now since $W^{h0}, W^{h1} \geq 0_I$ and $(S^{h1} - S^{h0}) \geq 0_I$ we get,

$$\frac{u^{h0}}{u^{h1}} \leq \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0})}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} \tag{A85}$$

and hence,

$$(1/2)(\Upsilon - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \geq (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \tag{A86}$$

where Υ is defined as,

$$\Upsilon \equiv \frac{p_N^1(p^0 \cdot c^{h0}) + p_N^0(p^1 \cdot c^{h0})}{p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1})} \tag{A87}$$

This in turn implies that,

[26]Note that $p_N^1(p^0 \cdot c^{h1}) + p_N^0(p^1 \cdot c^{h1}) \neq 0$ as $c^{h1} > 0_N$ and $p \gg 0_N$.

$$(1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^0}{p_N^0}) + (1/2)G^h(S^{h0}, S^{h1}, u^{h0}, \frac{p^1}{p_N^1}) =$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\frac{u^{h0}}{u^{h1}} - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} \geq$$

$$(1/2)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot [c^{h0} - c^{h1}] - (1/2)(\Upsilon - 1)[(\frac{p^0}{p_N^0}) + (\frac{p^1}{p_N^1})] \cdot c^{h1} = 0 \quad \text{(A88)}$$

where the last equality is derived after substituting for Υ (equation (A87)) and preforming some algebraic manipulations. □